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T. Hayashi,

M. Fujiwara, T. Kubota,

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Y. Okada and T. Takasu.

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On a Diophantine Equation,

by

KYÔICHI ÔISHI, Kagoshima.

Mr. T. Takenouchi(1) has recently made a discussion on a certain Diophantine equation and has proposed a very hard problem in his conclusive remark. The present paper is concerned with that problem, attacking only a very small part.

1. Consider a Diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots\ldots+\frac{1}{x_n}=\frac{b}{a}, \tag{1}$$

where $x_1, x_2, \ldots, x_n$ denote unknown positive integers, a and b known positive integers relatively prime to each other and $a > b$.

In the division of a by b, let the quotient increased by unity and the divisor decreased by the remainder be α_1 and r_1 respectively; and let those, in the division of $a\alpha_1$ by r_1, be α_2 and r_2 respectively; and let those, in the division of $a\alpha_1\alpha_2$ by r_2, be α_3 and r_3 respectively; repeating this we obtain the principal solution (2) $\alpha_1, \alpha_2, \ldots, \alpha_n$ of (1).

In the following we consider only the case $r_1 \neq 1$, $r_2=1$ (consequently $r_3=r_4=\ldots\ldots=1$), while Mr. Takenouchi considered the case $r_1=1$.

It can be seen at once that

$$\left.\begin{aligned}&\alpha_1=\left[\frac{a}{b}\right]+1, \quad \alpha_2=\left[\frac{a\alpha_1}{r_1}\right]+1,\\ &\alpha_k=a\alpha_1\alpha_2\ldots\ldots\alpha_{k-1}+1, \quad (k=3, 4, \ldots\ , n-1)\\ &\alpha_n=a\alpha_1\ldots\ldots\alpha_{n-1},\end{aligned}\right\} \tag{2}$$

and that $$\alpha_1 < \alpha_2 < \ldots\ldots < \alpha_n.$$

The following two natures are also to be noticed.

Between the principal solution $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ and a solution $x_1, x_2, \ldots\ldots, x_n$, there exist such relations that (3)

(1) Proceedings of the Math. Phys. Society of Japan, (3) 3 (1921).

(2) ibid., p. 81.

(3) ibid., pp. 80, 82.

1° $x_1=\alpha_1+h_1,\ x_2=\alpha_2'+h_2,\ x_3=\alpha_3''+h_3, \ldots,\ x_{n-1}=\alpha_{n-1}^{(n-2)}+h_{n-1},$
$x_n=\alpha_n^{(n-1)}, \qquad h_i\geqq 0\ (i=1, 2, \ldots).$

2°
$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\ldots+\frac{1}{\alpha_n}=\frac{b}{a},$$
$$\frac{1}{x_1}+\frac{1}{\alpha_2'}+\frac{1}{\alpha_3'}+\ldots+\frac{1}{\alpha_n'}=\frac{b}{a},$$
$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{\alpha_3''}+\ldots+\frac{1}{\alpha_n''}=\frac{b}{a},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{\alpha_n^{(n-1)}}=\frac{b}{a},$$

where $\alpha_{i+1}^{(i)}, \alpha_{i+2}^{(i)}, \ldots, \alpha_n^{(i)}$ are a principal set of $\dfrac{b}{a}-\dfrac{1}{x_1}-\ldots-\dfrac{1}{x_i}$.

To find the greatest possible value of x satisfying (1), we have to consider only those solutions in which no two x's are equal.

2. We are going to prove the

Theorem. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the principal solution of the equation (1) where r_2 defined in § 1 is equal to unity, and $x_1, x_2, \ldots, x_n$ any other solution such that $x_1<x_2<\ldots<x_n$, then x_n is less than α_n.

In the present section, we treat of the case where none of h's vanishes. We shall require a few lemmas to start with.

Lemma 1. If α and β be the first and second elements of the principal set of $\dfrac{B}{A}$, and β' be the first element of that of $\dfrac{B}{A}-\dfrac{1}{\alpha+h}$ $(h\neq 0)$, then
$$\beta' < \frac{1+h\alpha+\beta}{1+h}, \text{ and } \beta'<\beta.$$

Let R_1, R_2 and R_2' denote the pseudo remainders (i. e. divisor decreased by the remainder) in the divisions of A by B, $A\alpha$ by R_1, and $A(\alpha+h)$ by R_1+Bh respectively, then we have
$$A(\alpha+h)=(R_1+Bh)\beta'-R_2'<(R_1+Bh)\beta'.$$
Therefore β' is equal to the least integral value of β'' which satisfies the inequality
$$A\alpha-R_1\beta''<h(B\beta''-A), \text{ or } \frac{A\alpha}{R_1}-\beta''<h\frac{B\beta''-A}{R_1}.$$
Now since
$$\frac{A\alpha}{R_1}-\beta''<\beta-\beta'', \text{ and } \frac{B\beta''-A}{R_1}=\frac{B\beta''-B\alpha+R_1}{R_1}\geqq\beta''-\alpha+1,$$

it follows that β' is less than or equal to the least integral value of β'' which satisfies the inequality

$$\beta-\beta'' \leqq h(\beta''-\alpha+1), \text{ or } \frac{-h+h\alpha+\beta}{1+h} \leqq \beta''.$$

Hence

$$\beta' < \frac{-h+h\alpha+\beta}{1+h}+1=\frac{1+h\alpha+\beta}{1+h} \leqq \beta.$$

Lemma 2. Let $a_k = ax_1x_2\ldots x_k$, and $\dfrac{b}{a}-\dfrac{1}{x_1}-\ldots\ldots-\dfrac{1}{x_k}=\dfrac{b_k}{a_k}$,

then
$$\frac{a_k}{b_k} < \frac{a_{k-1}}{b_{k-1}} \cdot \frac{\alpha_k^{(k-1)}+h_k}{h_k}.$$

For the proof of this we may refer to Mr. Takenouchi's paper, l.c. pp. 83–84.

Now we turn to the theorem.

Putting $A=a$, $B=b$ in lemma 1, we get $\alpha_2' < \alpha_2$. And this gives two results:

$$\frac{a_1}{b_1} < \frac{a\alpha_1}{r_1}, \text{ and } \frac{\alpha_2'+h_2}{h_2} \leqq \alpha_2.$$

Also we have, by lemma 2, that

$$\frac{a_2}{b_2} < \frac{a_1}{b_1} \cdot \frac{\alpha_2'+h_2}{h_2}.$$

Hence from the last three inequalities, we get

$$\frac{a_2}{b_2} < \frac{a\alpha_1\alpha_2}{r_1}, \text{ consequently } \alpha_3'' < \alpha_3.$$

By repeating the similar argument as we get $\alpha_3'' < \alpha_3$ from the fact $\alpha_2' < \alpha_2$, we can show $\alpha_n^{(n-1)} < \alpha_n$, i.e. $x_n < \alpha_n$.

3. Dealing with the case, where some of h's vanish, Mr. Takenouchi has tried to give the proof of

$$\alpha_{k+1}^{(k)} < \alpha_{k+1} \tag{1}$$

provided $h_k=0$. To make sure of (1), he has intended to show that

$$a_{k-1}\,\alpha_k^{(k-1)} < a\alpha_1\alpha_2\ldots\alpha_k, \tag{2}$$

provided $h_k=0$, since (2) is a sufficient condition for (1). Again, to make sure of (2), he has shown that, except some special cases, there are two natures:

If $h_2=0$, then $a_1\alpha_2' < a\alpha_1\alpha_2$. (3)

If $h_k=0$, and $a_{i-2}\alpha_{i-1}^{(i-2)} < a\alpha_1\alpha_2\ldots\alpha_{i-1}$ $(i=3, 4, \ldots k)$, (4)

then $$a_{k-1}\alpha_k^{(k-1)} < a\alpha_1\alpha_2\ldots.\alpha_k.$$

And he has concluded adventurously that, from (3) and (4), it always follows (2), and consequently (1), except the special cases.

I dared to say "adventurously," for there are many cases, besides the above exceptional ones, in which we can not apply (3) and (4), and consequently (2) can never be deduced from this point of view.

I take an example to explain this.

Consider $a=7,\ b=1,\ h_1=43.$

Then we have $$\alpha_1=8,\ \alpha_2=57,\ a_1=7\cdot 51,\ b_1=44,\ \alpha_2'=9,$$

since $$\frac{b}{a}-\frac{1}{\alpha_1+h_1}=\frac{b_1}{a_1},\quad \alpha_2'=\left[\frac{a_1}{b_1}\right]+1.$$

Hence $$a_1\alpha_2'=7\cdot 51\cdot 9,\quad a\alpha_1\alpha_2=7\cdot 8\cdot 57,\quad a_1\alpha_2'>a\alpha_1\alpha_2.$$

The last inequality does not satisfy the assumption of (4), and we can not apply (4). It is evident that we can not apply (3), and that this case is not involved in the exceptional cases, in which he has given a full discussion and which are indeed

(i) $a=1$ or 2, $b=1$; (see p. 85)

(ii) $a\alpha_1\alpha_2\ldots.\alpha_{k-2}\leqq 12,\ \alpha_{k-1}^{(k-2)}\leqq 6\ (k\geqq 3).$ (see p. 87)

Moreover it may easily be seen that we can always consider k such as $h_k=0$ taking n sufficiently great. Hence this is a case which has never been considered at all by him.

I have thought whether some slight modification in his method enables us to show (2) or not, and have tried to replace (4) by (4′) and (4″):

If $h_k=0,\ h_{k-1}=0$ and $a_{k-2}\alpha_{k-1}^{(k-2)}<a\alpha_1\alpha_2\ldots.\alpha_{k-1}$,

then $$a_{k-1}\alpha_k^{(k-1)}<a\alpha_1\alpha_2\ldots.\alpha_k.\tag{4′}$$

If $h_k=0,\ h_{k-1}\neq 0$ and $a_{k-2}\alpha_{k-1}^{(k-2)}<a\alpha_1\alpha_2\ldots.\alpha_{k-1}$,

then $$a_{k-1}\alpha_k^{(k-1)}<a\alpha_1\alpha_2\ldots.\alpha_k.\tag{4″}$$

The above replacement can be done easily by reading through once his proof of (4).

But, to my disappointment I know that, only in case $h_1\neq 0$, $h_2=h_3=\ldots.=h_k=0$, we can deduce the conclusion (2), from (3) and (4′), and that it is perhaps impossible to apply (4″) unless some new discussion is developed. In fact, there can be found no simple way in the process of his original discussion to ascertain the assumption

$$a_{k-2}\alpha_{k-1}^{(k-2)}<a\alpha_1\alpha_2\ldots.\alpha_{k-1}\tag{5}$$

when $h_{k-1}\neq 0$, although we are sometimes able to ascertain

$$\alpha_k^{(k-1)}<\alpha_k\tag{6}$$

when $h_{k-1}\neq 0$, (6) being a necessary but not sufficient condition for (5). To make up for this defect we must develop some discussion.

Indeed, we can establish also our lemma 8 in case $r_1=1$, in a quite similar manner as in case $r_1\neq 1$, $r_2=1$. This lemma will serve to ascertain (5) and to make (4″) applicable. Thus we get always (2).

4. We proceed now to discuss the case, where some of h's vanish and $h_1\neq 0$; the case $h_1=0$ reduces to the problem already studied by Mr. Takenouchi. Thus, our object is to show that $\alpha_{k+1}^{(k)}<\alpha_{k+1}$ for every k, for which $h_k=0$ $(k\geqq 2)$.

This will be attained[1] by showing that

$$a_{k-1}\alpha_k^{(k-1)}<a\alpha_1\ldots\alpha_k$$

for every k for which $h_k=0(k\geqq 2)$, a_{k-1} having the same meaning as in lemma 2. To see the last fact we also require some lemmas.

Lemma 3. If $x_1\leqq\alpha_2'$ then $a_1\alpha_2'<a\alpha_1\alpha_2$.

The proof of the lemma is divided into four parts: [i] $\alpha_1\geqq 5$, [ii] $\alpha_1=4$, [iii] $\alpha_1=3$, [iv] $\alpha_1=2$. We shall first consider the case

[i] $\alpha_1\geqq 5$.

From the assumption, it follows that

$$\frac{1}{x_1}\geqq\frac{1}{\alpha_2'}=\frac{1}{\left[\dfrac{a_1}{b_1}\right]+1}\geqq\frac{1}{\dfrac{a_1}{b_1}+1}=\frac{\dfrac{b_1}{a_1}}{1+\dfrac{b_1}{a_1}}=\frac{\dfrac{b}{a}-\dfrac{1}{x_1}}{1+\dfrac{b}{a}-\dfrac{1}{x_1}}>\frac{\dfrac{b}{a}-\dfrac{1}{x_1}}{1+\dfrac{b}{a}}$$

or $$x_1<\frac{2a+b}{b}\text{ and consequently }x_1<2\alpha_1+1. \qquad (3)$$

On the other hand, a sufficient condition in order that $a_1\alpha_2'<a\alpha_1\alpha_2$ is that

$$a(\alpha_1+h_1)\left\{\frac{a(\alpha_1+h_1)}{bh_1+r_1}+1\right\}<a\alpha_1\alpha_2,$$

or $$\alpha_1+h_1<\frac{a\alpha_2}{a+b+\dfrac{r_1}{h_1}-\dfrac{1}{h_1}}.$$

Hence making use of the relations (3) and

$$\frac{a}{a+b+\dfrac{r_1}{h_1}-\dfrac{1}{h_1}}>\frac{a}{a+2b},$$

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(1) For, $\alpha_{k+1}^{(k)}=\left[\dfrac{a_k}{b_k}\right]+1=\left[\dfrac{a_{k-1}(\alpha_k^{(k-1)}+h_k)}{b_k}\right]+1=\left[\dfrac{\alpha_{k-1}\alpha_k^{(k-1)}}{b_k}\right]+1,$ and $\alpha_{k+1}=a\alpha_1\ldots\alpha_k+1$ $(k\geqq 2)$.

it is obvious that a sufficient condition in order that $a_1\alpha_2' < a\alpha_1\alpha_2$ is that

$$\frac{2a+b}{b} \leqq \frac{a\alpha_2}{a+2b} \quad \text{or} \quad \frac{2a}{b}+5+\frac{2b}{a} \leqq \alpha_2 .$$

The last inequality is satisfied when $\alpha_1 \geqq 5$, for we have from (2) that

$$\frac{a}{b} < \alpha_1, \quad \frac{b}{a} < \frac{1}{\alpha_1 - 1} \text{ and } (\alpha_1-1)\alpha_1+1 \leqq \alpha_2 .$$

It now remains to consider the cases [ii], [iii] and [iv]. Each of them, however, may be treated by a method similar to one another, so that we shall here take the case [iv] only.

[iv] $\alpha_1 = 2$.

In consequence of lemma 1, we get

$$\alpha_2' < \frac{1+2h_1+\alpha_2}{1+h_1} .$$

Hence we shall show that the inequality

$$(2+h_1)\frac{1+2h_1+\alpha_2}{1+h_1} \leqq 2\alpha_2 \tag{4}$$

will hold good all the time except some trivial cases, instead of showing $a_1\alpha_2' < a\alpha_1\alpha_2$.

Now from (3) we have $2+h_1 = x_1 < 2\alpha_1+1 = 5$. Hence $h_1 = 1$ or 2. When $h_1=1$, the inequality (4) which we propose to prove becomes

$$3 . \frac{3+\alpha_2}{2} \leqq 2\alpha_2, \text{ or } 9 \leqq \alpha_2 .$$

Therefore there are two trivial cases when the above inequality is not satisfied, viz.

$$\alpha_2 = 5,\ 7,$$

observing(1) that, (1°) α_1 and α_2 are relatively prime to each other, (2°) $\alpha_2 \geqq \alpha_1(\alpha_1-1)+1 = 3$, (3°) $3 = \alpha_1 + h_1 = x_1 \leqq \alpha_2' < \dfrac{1+2h_1+\alpha_2}{1+h_1} = \dfrac{3+\alpha_2}{2}$.

If $\alpha_2 = 5$, then $\alpha_2' < (3+\alpha_2) \div 2 = 4$, i.e. $\alpha_2' \leqq 3$. Hence

$$a_1\alpha_2' = a(\alpha_1+h_1)\alpha_2' \leqq a{\cdot}3{\cdot}3 < a{\cdot}2{\cdot}5 = a\alpha_1\alpha_2 .$$

If $\alpha_2 = 7$, then $\alpha_2' < (3+\alpha_2) \div 2 = 5$, i.e. $\alpha_2' \leqq 4$. Hence

$$a_1\alpha_2' = a(\alpha_1+h_1)\alpha_2' \leqq a{\cdot}3{\cdot}4 < a{\cdot}2{\cdot}7 = a\alpha_1\alpha_2 .$$

(1) The nature 1° follows from the fact that r_2 defined in §1 equals to unity; 2° follows from (2); and 3° follows from the assumption and lemma 1. In this case 2° is of no use. It becomes however useful in other cases.

Similarly when $h_1=2$, we get $a_1\alpha_2' < a\alpha_1\alpha_2$.

Lemma 4. If 1° $x_{k-1} \leqq \alpha_k^{(k-1)}$; 2° $a_{k-2}\alpha_{k-1}^{(k-2)} \leqq a\alpha_1\alpha_2\ldots\alpha_{k-1}$ and $\alpha_{k-1}^{(k-2)} \leqq \alpha_{k-1}$, signs of equality standing only when $k=2$; then $a_{k-1}\alpha_k^{(k-1)} < a\alpha_1\alpha_2\ldots\alpha_k$.

When $k=2$, this becomes lemma 3. We may therefore suppose that $k \geqq 3$. The proof of this lemma is also divided into three parts: [i] $\alpha_{k-1}^{(k-2)} \geqq 4$, [ii] $\alpha_{k-1}^{(k-2)}=3$, [iii] $\alpha_{k-1}^{(k-2)}=2$. We shall first consider the case [i] $\alpha_{k-1}^{(k-2)} \geqq 4$.

Since $a_{k-1}\alpha_k^{(k-1)} \equiv a_{k-2}x_{k-1}\alpha_k^{(k-1)} \leqq a_{k-2}(\alpha_{k-1}^{(k-2)}+h_{k-1})\left\{\dfrac{a_{k-2}(\alpha_{k-1}^{(k-2)}+h_{k-1})}{b_{k-2}h_{k-1}+1}+1\right\}$

$$< a_{k-2}(\alpha_{k-1}^{(k-2)}+h_{k-1})\{\alpha_k+h_{k-1}(a_{k-2}+b_{k-2})\}\frac{1}{b_{k-2}h_{k-1}+1}, \text{ (by 2° \&(2))}$$

the sufficient condition in order that $a_{k-1}\alpha_k^{(k-1)} < a\alpha_1\alpha_2\ldots\alpha_k$ is that

$$a_{k-2}(\alpha_{k-1}^{(k-2)}+h_{k-1})\{\alpha_k+h_{k-1}(a_{k-2}+b_{k-2})\} \leqq a\alpha_1\alpha_2\ldots\alpha_k(b_{k-2}h_{k-1}+1)$$

or
$$\alpha_k(a_{k-2}\alpha_{k-1}^{(k-2)}-a\alpha_1\ldots\alpha_{k-1})$$
$$+h_k\{a_{k-2}\alpha_k+a_{k-2}(\alpha_{k-1}^{(k-2)}+h_{k-1})(a_{k-2}+b_{k-2})-a\alpha_1\ldots\alpha_k b_{k-2}\} \leqq 0.$$

Again since by 2° the quantity in the first parenthesis is negative, the sufficient condition becomes that the quantity in braces is not positive, i.e.

$$\alpha_{k-1}^{(k-2)}+h_{k-1} \leqq \frac{(a\alpha_1\ldots\alpha_{k-1}b_{k-2}-a_{k-2})\alpha_k}{a_{k-2}(a_{k-2}+b_{k-2})}$$

$$=\frac{\alpha_k^2 b_{k-2}}{a_{k-2}(a_{k-2}+b_{k-2})}-\frac{\alpha_k}{a_{k-2}}. \quad \text{(by (2))}$$

Now the supposition $\alpha_{k-1}^{(k-2)} \geqq 4$ gives rise to the relation $a_{k-2} > 3b_{k-2}$, and so $a_{k-2}+b_{k-2} < \frac{4}{3}a_{k-2}$; and the assumption 1° gives rise to the fact

$$\alpha_{k-1}^{(k-2)}+h_{k-1} < \frac{2a_{k-2}+b_{k-2}}{b_{k-2}} \text{ (}^1\text{)} < \frac{7}{3}\frac{a_{k-2}}{b_{k-2}}.$$

Hence the sufficient condition becomes that

$$\frac{7}{3}\frac{a_{k-2}}{b_{k-2}} \leqq \frac{\alpha_k}{a_{k-2}}\left(\frac{3}{4}\frac{\alpha_k b_{k-2}}{a_{k-2}}-1\right),$$

or
$$a_{k-2} \leqq \frac{3}{7}\frac{\alpha_k b_{k-2}}{a_{k-2}}\left(\frac{3}{4}\frac{\alpha_k b_{k-2}}{a_{k-2}}-1\right).$$

Further, from the second part of 2°, we have

$$\frac{b_{k-2}}{a_{k-2}} \geqq \frac{1}{\alpha_{k-1}^{(k-2)}} > \frac{1}{\alpha_{k-1}},$$

---

(1) This can be seen in the same manner as (3) in lemma 3 was seen.

and so $$\frac{\alpha_k b_{k-2}}{a_{k-2}} > \frac{\alpha_k}{\alpha_{k-1}} = \frac{a\alpha_1 \ldots \alpha_{k-1}+1}{\alpha_{k-1}} > a\alpha_1 \ldots \alpha_{k-2}.$$

From the first part of 2° it follows that, when $k \geqq 4$,

$$a_{k-2} < \frac{a\alpha_1 \ldots \alpha_{k-1}}{\alpha_{k-1}^{(k-2)}} = \frac{1}{\alpha_{k-1}^{(k-2)}} a\alpha_1 \ldots \alpha_{k-2}(a\alpha_1 \ldots \alpha_{k-1}+1),$$

and when $k=3$,

$$a_1 < \frac{a\alpha_1\alpha_2}{\alpha_2'} \leqq \frac{1}{\alpha_2'} a\alpha_1\left(\frac{a\alpha_1}{2}+1\right).$$

Therefore the sufficient condition becomes that, when $k \geqq 4$,

$$\frac{1}{\alpha_{k-1}^{(k-2)}} a\alpha_1 \ldots \alpha_{k-2}(a\alpha_1 \ldots \alpha_{k-2}+1) \leqq \frac{3}{7} a\alpha_1 \ldots \alpha_{k-2}\left(\frac{3}{4} a\alpha_1 \ldots \alpha_{k-2}-1\right),$$

and when $k=3$, $$\frac{a\alpha_1}{\alpha_2'}\left(\frac{a\alpha_1}{2}+1\right) \leqq \frac{3}{7} a\alpha_1\left(\frac{3}{4} a\alpha_1-1\right).$$

The former inequality is satisfied by all values of $a\alpha_1 \ldots \alpha_{k-2}$ which are not less than $a\alpha_1\alpha_2 > 18$, and of $\alpha_{k-1}^{(k-2)} \geqq 4$, while the latter, by all values of $a\alpha_1$ which are not less than 6 and of $\alpha_{k-1}^{(k-2)} \geqq 4$.

[ii] $\alpha_{k-1}^{(k-2)}=3$.

From the assumption 1° it follows that

$$3+h_{k-1}=x_{k-1} < \frac{2a_{k-2}+b_{k-2}}{b_{k-2}} < 2\alpha_{k-1}^{(k-2)}+1=7.$$

Hence we have $h_{k-1}=0, 1, 2$ or 3.

First suppose that $h_{k-1}=0$, then it is clear by the first part of the assumption 2° that we have $a_{k-1}\alpha_k^{(k-1)} < a\alpha \ldots \alpha_k$.

Secondly suppose that $h_{k-1}=1$, then we have three cases:—

Case 1. $x_1=2$, $x_2=3$, $x_3=4$; $k=4$,

Case 2. $x_1=2$, $x_k=4$; $k=3$,

Case 3. $x_1=3$, $x_k=4$; $k=3$.

In cases 1 and 2 we have $h_1=0$, which is contrary to our supposition stated at the beginning of this section.

In case 3; we have $\alpha_1=2$, $h_1=1$. Now by lemma 1 we get

$$3=\alpha_2' < \frac{1+h_1\alpha_1+\alpha_2}{1+h_1} = \frac{3+\alpha_2}{2}, \text{ i.e. } 5 \leqq \alpha_2 \ (^1);$$

and $$\alpha_3'' < \frac{1+h_2\alpha_2'+\alpha_3'}{1+h_2} = \frac{4+\alpha_3'}{2} \leqq \frac{3+\alpha_3}{2} \ (^2).$$

(1) α_1 and α_2 are relatively prime numbers. Hence $\alpha_2 \neq 4$.

(2) The fact $\alpha'_3 < \alpha_3$ can be seen from the first part of the assumption 2°.

Therefore $a_2\alpha_3'' = ax_1x_2\alpha_3'' < a\cdot3\cdot4\cdot\dfrac{3+\alpha_3}{2} < a\cdot2\cdot4\alpha_3$[1] $\leqq a\alpha_1\alpha_2\alpha_3$.

In a precisely similar manner we have $a_{k-1}\alpha_k < a\alpha_1\ldots\alpha_k$, supposing $h_{k-1}=2$ or 3.

[iii] $\alpha_{k-1}^{(k-2)}=2$.

The similar argument as above may be applied too.

Lemma 5. If $x_1>\alpha_2'$, $x_2>\alpha_3''$, ..., $x_{k-2}>\alpha_{k-1}^{(k-2)}$, $x_{k-1}=\alpha_k^{(k-1)}$; then $a_{k-1}\alpha_k^{(k-1)}<a\alpha_1\ldots\alpha_k$.

When $k=2$, this agrees with a part of lemma 3. We may therefore suppose that $k\geqq3$. The proof of the lemma is divided into many parts according to the value of α_1.

First of all we suppose $\alpha_1\geqq5$.

We notice here four natures which are immediate consequences of the assumptions:

(*a*) $h_i\geqq2$, $i=2, 3, \ldots, k-1$.

(*b*) $\alpha_1\geqq5$, $\alpha_2\geqq\alpha_1(\alpha_1-1)+1\geqq21$, $\alpha_3\geqq\alpha_2(\alpha_2-1)+1\geqq421$,

and $\alpha_i>(\alpha_2-1)2^{i-2}$ $i=3, 4, 5, \ldots$

(*c*) $\alpha_2'\leqq2\alpha_1-1$.

(*d*) $x_{k-2}<\dfrac{3a_{k-3}}{b_{k-3}}$, $x_{k-4}<\dfrac{4a_{k-4}}{b_{k-4}}$, ..., $x_1<\dfrac{ka}{b}$.

(*a*) is a consequence of the relations

$$x_{i-1}<x_i=\alpha_i^{(i-1)}+h_i \text{ and } x_{i-1}>\alpha_i^{(i-1)}, \quad (i=2, 3, \ldots, k-1)$$

and (*b*) is that of the assumption $\alpha_1\geqq5$ and the formula (2).

(*c*) can be seen from the following consideration: When h_1 is supposed to increase from zero to infinity, α_2' will decrease from α_2 to α_1. Besides, when $\alpha_1+h_1<\alpha_2'$, then $\alpha_1+h_1\leqq\dfrac{2a}{b}$; and when $\alpha_1+h_1\geqq\alpha_2'$, then $\alpha_1+h_1>\dfrac{2a}{b}$. Hence if we trace two curves $y=f(h_1)\equiv\alpha_1+h_1$ and $y=\varphi(h_1)\equiv\alpha_2'$, and pay attention to the point of intersection, it can readily be seen that $\alpha_2'\leqq\dfrac{2a}{b}$, or $\alpha_2'\leqq2\alpha_1-1$.

(*d*) follows from the facts: $x_{k-2}<\alpha_k^{(k-1)}$, $x_{k-3}(<x_{k-2})<\alpha_k^{(k-1)}$, etc. Indeed, for example, take a fact $x_{k-4}<\alpha_k^{(k-4)}$, then we have

$$\frac{1}{x_{k-4}}\geqq\frac{1}{\alpha_k^{(k-1)}-1}=\frac{1}{\left[\dfrac{a_{k-1}}{b_{k-1}}\right]}\geqq\frac{b_{k-1}}{a_{k-1}}$$

---

(1) This comes from the fact $\alpha_3\geqq\alpha_2(\alpha_2-1)+1\geqq21$.

$$=\frac{b_{k-5}}{a_{k-5}}-\frac{1}{x_{k-4}}-\frac{1}{x_{k-3}}-\frac{1}{x_{k-2}}-\frac{1}{x_{k-1}}>\frac{b_{k-5}}{a_{k-5}}-\frac{4}{x_{k-4}},$$

and from this follows the relation $x_{k-4}<\frac{5a_{k-5}}{b_{k-5}}$.

We now proceed to show the lemma. At first let us consider the case $k=3$, and prove that $ax_1x_2\alpha_3''<a\alpha_1\alpha_2\alpha_3$.

From (d) and the assumption $x_2=\alpha_3''$, we have

$$ax_1x_2\alpha_3''<a\frac{3a}{b}\alpha_3''\alpha_3''.$$

Also we have a set of relations:—

$$\frac{a}{b}<\alpha_1,$$

$$\frac{a_1}{b_1}<\alpha_2'\leqq 2\alpha_1-1, \qquad \text{(by } (c))$$

$$\frac{a_2}{b_2}<\frac{a_1}{b_1}\frac{\alpha_2'+h_2}{h_2}<\frac{a_1}{b_1}\frac{\alpha_2'+2}{2}<\alpha_2'\frac{\alpha_2'+2}{2} \qquad \text{(by lemma 2 und } (a))$$

$$<(2\alpha_1-1)(\alpha_1+1)\leqq\frac{54\alpha_2}{21} \qquad \text{(by } (b))$$

$$\alpha_3''\equiv\left[\frac{a_2}{b_2}\right]+1\leqq\frac{54\alpha_2}{21}\leqq\frac{54\alpha_3}{421}. \qquad \text{(by } (b))$$

Hence we obtain

$$ax_1x_2\alpha_3''<a\cdot\frac{3a}{b}\alpha_3''\alpha_3''<a\cdot 3\alpha_1\cdot\frac{54}{21}\alpha_2\frac{54}{421}\alpha_3<a\alpha_1\alpha_2\alpha_3.$$

Next let us consider the general case $k\geqq 4$, and prove that

$$ax_1\ldots x_{k-1}\alpha_k^{(k-1)}<a\alpha_1\ldots\alpha_k.$$

In fact, we have as before a set of relations:

$$\frac{a}{b}<\alpha_1,$$

$$\frac{a_1}{b_1}<2\alpha_1-1\leqq\frac{9\alpha_2}{21}, \qquad \alpha_2'\leqq 2\alpha_1-1,$$

$$\frac{a_2}{b_2}<\frac{54\alpha_2}{21}<4\alpha_2-6, \qquad \alpha_3''\leqq 4\alpha_2-6,$$

$$\frac{a_3}{b_3}<\frac{a_2}{b_2}\frac{\alpha_3''+h_3}{h_3}\leqq\frac{a_2}{b_2}\frac{\alpha_3''+2}{2}<(4\alpha_2-6)(2\alpha_2-2),$$

$$\alpha_4'''\leqq(4\alpha_2-6)(2\alpha_2-2),$$

$$\frac{a_4}{b_4} < \frac{a_3}{b_3} \frac{\alpha_4'' + 2}{2} < \frac{a_3}{b_3} \frac{\alpha_4''' + 2(2\alpha_2 - 2)}{2} < (4\alpha_2 - 6)(2\alpha_2 - 2)^3,$$

$$\alpha_5^{(4)} \leqq (4\alpha_2 - 6)(2\alpha_2 - 2)^3,$$

. . . .

$$\frac{a_i}{b_i} < \frac{a_{i-1}}{b_{i-1}} \frac{\alpha_i^{(i-1)} + 2(2\alpha_2 - 2)^{2^{i-3}-1}}{2} < (4\alpha_2 - 6)(2\alpha_2 - 2),^{2^{i-2}-1}$$

$$\alpha_{i+1}^{(i)} \leqq (4\alpha_2 - 6)(2\alpha_2 - 2),^{2^{i-2}-1}$$

. . . .

Consequently we obtain a set of inequalities:

$$\frac{a}{b} < \alpha_1, \qquad \frac{a_1}{b_1} < \frac{1}{2}\alpha_2, \qquad \frac{a_i}{b_i} < 2(2\alpha_2 - 2)^{2^{i-2}}, \ (i = 2, 3, \ldots, k-3)$$

$$\alpha_k^{(k-1)} < 2(2\alpha_2 - 2)^{2^{k-3}}.$$

Also, from (d) and the assumption $x_{k-1} = \alpha_k^{(k-1)}$, we have

$$a x_1 x_2 \ldots x_{k-1} x_k^{(k-1)} < \frac{\lfloor k}{2} \cdot \frac{1}{2} \cdot 2^{k-2} \cdot a\alpha_1 \alpha_2 (2\alpha_2 - 2)^{\sum_{i=2}^{k-3} 2^{i-2} + 2.2^{k-3}}$$

$$= \lfloor k \; 2^{5.2^{k-4}+k-5} \cdot a\alpha_1 \alpha_2 (\alpha_2 - 1)^{5.2^{k-4}-1},$$

On the other hand, from (b) we have

$$a\alpha_1 \alpha_2 \ldots \alpha_k > a\alpha_1 \alpha_2 (\alpha_2 - 1)^{\sum_{i=3}^{k} 2^{i-2}} = a\alpha_1 \alpha_2 (\alpha_2 - 1)^{2^{k-1}-2}.$$

Therefore a sufficient condition in order that $a x_1 \ldots x_{k-1} \alpha_k^{(k-1)} < a\alpha_1 \ldots \alpha_k$ is that

$$a\alpha_1 \alpha_2 (\alpha_2 - 1)^{2^{k-1}-2} > \lfloor k \; 2^{5.2^{k-4}+k-5} \cdot a\alpha_1 \alpha_2 (\alpha_2 - 1)^{5.2^{k-4}-1},$$

or
$$(\alpha_2 - 1)^{3.2^{k-4}-1} > \lfloor k \cdot 2^{5.2^{k-4}+k-5}.$$

Since $\alpha_2 - 1 > 2^4$, the sufficient condition becomes that

$$(2^4)^{3.2^{k-4}-1} > \lfloor k \; 2^{5.2^{k-4}+k-5}$$

or
$$2^{7.2^{k-4}-k+1} > \lfloor k \;, \ (k \geqq 4).$$

And we can see, by mathematical induction, that the last inequality is really true.

The remaining cases i.e. $\alpha_1 = 4$, 3 or 2, can be treated by an analogous reasoning as the case $\alpha_1 \geqq 5$ in the present lemma and the cases (ii), (iii), etc. in the lemma 3, 4 were done.

The similar treatment as before will also enable us to establish

Lemma 6. If $x_1 > \alpha_2'$, $x_2 > \alpha_3''$, ..., $x_{k-2} > \alpha_{k-1}^{(k-2)}$, $x_{k-1} < \alpha_k^{(k-1)}$, then $a_{k-1}\alpha_k^{(k-1)} < a\alpha_1 \dots . \alpha_k$.

Combining lemmas 3, 4, 5 and 6, we have

Lemma 7. If $x_1 > \alpha_2'$, $x_2 > \alpha_3''$, ..., $x_{k-2} > \alpha_{k-1}^{(k-2)}$, $x_{k-1} \leqq \alpha_k^{(k-1)}$, then $\quad a_{k-1}\alpha_k^{(k-1)} < a\alpha_1 \dots . \alpha_k$. $(k \geqq 2)$

4. Lemma 8. If (1°) $x_k > \alpha_{k+1}^{(k)}$, $x_{k+1} > \alpha_{k+2}^{(k+1)}$, ..., $x_{k+l-1} \leqq \alpha_{k+l}^{(k+l-1)}$; (2°) $a_{k-1}\alpha_k^{(k-1)} \leqq a\alpha_1, \dots . \alpha_k$, and $\alpha_k^{(k-1)} \leqq \alpha_k$, equality signs standing only when $k = l = 1$; then $a_{k+l-1}\alpha_{k+l}^{(k+l-1)} < a\alpha_1 \dots . \alpha_{k+l}$ $(k, l = 1, 2, \dots .)$.

When $k=1$, this is nothing other than the lemma 7. Hence we may suppose $k \geqq 2$.

The proof of this lemma is so similar to that of the case $\alpha_1 \geqq 5$ in lemma 5, that it is hardly necessary to write it out at length. We have only to observe the following.

Since $$x_k < \frac{(l+1)a_{k-}}{b_{k-1}}\text{(1)} < (l+1)\alpha_k^{(k-1)}$$

we have $$ax_1 \dots . x_k = a_{k-1}\alpha_k x < (l+1)a_{k-1}\alpha_k^{(k-1)}$$
$$< (l+1)a\alpha_1 \dots . \alpha_k. \quad \text{(by 2°)}$$

Hence there exist four properties analogous to (a), (b), (c), (d) in lemma 5, viz.

(a) $h_i \geqq 2$, $i = k+1, k+2, \dots ., k+l-1$,

(b) $\alpha_k \geqq 5$,(2) $\alpha_{k+1} \geqq 21$, $\alpha_{k+2} \geqq 421$, and $\alpha_i > (\alpha_{k+1}-1)2^{i-2}$, $(i = k+2, k+3, \dots .)$,

(c) $\alpha_{k+1}^{(k)} < 2\alpha_k^{(k-1)} - 1 < 2\alpha_k - 1$, (by the second part of 2°)

(d) $x_{k+l-2} < \dfrac{3a_{k+l-2}}{b_{k+l-3}}$, $x_{k+l-3} < \dfrac{4a_{k+l-4}}{b_{k+l-4}}$, ..., $x_{k+} < \dfrac{la_k}{b_k}$,

$ax_1 \dots . x_k < (l+1).a\alpha_1 \dots . \alpha_k$.

We can get the result by the same method which we applied in the proof of lemma 5.

5. We are now able to prove that

$$a_{k-1}\alpha_k^{(k-1)} < a\alpha_1 \dots . \alpha_k \text{ provided } h_k = 0 \ (k \geqq 2). \tag{5}$$

For, let h_k' be the first one which vanishes. Then we shall have

$$a_{k'-1}\alpha_{k'}^{(k'-1)} < a\alpha_1 \dots . \alpha_{k'}. \tag{6}$$

Indeed, let us suppose that two sets of numbers $x_1, x_2, \dots ., x_{k'}$ and $\alpha_1, \alpha_2', \dots ., \alpha_{k'}^{(k'-1)}$ are, for example, such that

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(1) This inequality follows from the fact $x_k < \alpha_{k+l}^{(k+l-1)}$.

(2) If $\alpha_k < 5$, then $\alpha_1 = 2$, $\alpha_2 \equiv \alpha_k = 3$, since α_1 and α_2 are relatively prime numbers. From 2°, we have $\alpha_2 > \alpha_2' = 2$ and $a_1\alpha_2' = ax_1\alpha_2' < a\alpha_1\alpha_2 = 6a$. Hence $\alpha_1 + h_1 = x_1 = 2$, which is inadmissible as we suppose $h_1 \neq 0$.

$$x_1 > \alpha_2', \ x_2 > \alpha_3'', \ x_3 = \alpha_4''', \ x_4 < \alpha_5^{(4)}, \ \ldots, \ x_{k'-1} < \alpha_{k'}^{(k'-1)}.$$

First apply the lemma 8, putting there $k=1$, $l=3$; we have then

$$\alpha_5 \alpha_4''' < a\alpha_1 \ \ldots \alpha_4.$$

On the other hand, we have, in the same manner as we treated the case when none of h's vanishes in § 2,

$$\alpha_4''' < \alpha_4.$$

The last two inequalities are useful in the next application of the same lemma, since they satisfy the assumption 2° when $k=4$.

Next apply the lemma, putting $k=4$, $l=1$; we have then $a_4\alpha_5^{(4)} < a\alpha_1 \ldots \alpha_5$. Also we have $\alpha_5^{(4)} < \alpha_5$. These two inequalities shall serve to the next application of the lemma; and we repeat this process till we arrive at the results

$$a_{k'-1}\alpha_{k'}^{(k'-1)} < a\alpha_1 \ldots . \alpha_{k'}, \tag{6}$$

and consequently $\alpha_{k'+1}^{(k')} < \alpha_{k'+1}$. (7)

Let h_x'' be the second one which vanishes, then we have, as preceding, the result, corresponding to (6), which is a part of (5). It is to be noticed that, in this case $x_{k'} = \alpha_{k'}^{(k'-1)} + h_{k'} = \alpha_{k'}^{(k'-1)}$, so that from (6) and (7) we have

$$a_{k'}\alpha_{k'+1}^{(k)} < a\alpha_1 \ldots . \alpha_{k'+1},$$

and

$$\alpha_{k'+1}^{(k')} < \alpha_{k'+1}. \tag{7}$$

These last two inequalities shall serve to the applicability of the lemma.

In this way we have the result (5).

Thus we have completed the proof of the theorem stated in § 2.

Sur le champ de gravitation dans l'espace vide,

par

KINNOSUKE OGURA, à Paris.

Le but de ce Mémoire est d'étudier quelques problèmes sur le champ (statique sauf que les paragraphes 10, 11 et 12) de gravitation dans l'espace vide. Comme loi de gravitation pour l'espace-temps

$$ds^2 = \Sigma g_{\mu\nu} dx_\mu \, dx_\nu \quad (\mu, \nu = 1, 2, 3, 4),$$

je prends

$$R_{\mu\nu} \equiv \sum_\alpha \frac{\partial}{\partial x_\alpha} \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} - \sum_{\alpha,\beta} \begin{Bmatrix} \mu\alpha \\ \beta \end{Bmatrix} \begin{Bmatrix} \nu\beta \\ \alpha \end{Bmatrix} + \sum_\alpha \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} \frac{\partial}{\partial x_\alpha} \log \sqrt{-g} - \frac{\partial^2}{\partial x_\mu \partial x_\nu} \log \sqrt{-g}$$
$$= 0\ ;$$

mais je ne considère pas la loi modifiée :

$$R_{\mu\nu} = \lambda g_{\mu\nu}.$$

Quelques-uns des résultats ont été déjà communiqués à l'Académie des Sciences de Paris[1].

TABLE DES MATIÈRES.

Une méthode de l'intégration des équations différentiélles de gravitation.

1. Soit

$$ds^2 = f^2 dt^2 - d\sigma^2$$

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(1) Voir K. Ogura, Comptes Rendus, t. 173 (1921), p. 521, p. 641, p. 766.

$$=f^2dt^2-H_1^2dx_1^2-H_2^2dx_2^2-H_3^2dx_3^2$$

l'expression de l'intervalle élémentaire d'un champ *statique* dans un espace *vide*. Les fonctions f, H_1, H_2 et H_3 ne dépendent pas de t, et les tenseurs de Riemann-Christoffel contractés P_{11}, P_{12}, ...de la forme ternaire

$$d\sigma^2=H_1^2\,dx_1^2+H_2^2\,dx_2^2+H_3^2\,dx_3^2$$

deviennent

$$-P_{11}=H_1\left(\frac{Q_2}{H_3}+\frac{Q_3}{H_2}\right),\qquad -P_{22}=H_2\left(\frac{Q_3}{H_1}+\frac{Q_1}{H_3}\right),$$

$$-P_{33}=H_3\left(\frac{Q_1}{H_2}+\frac{Q_2}{H_1}\right),$$

$$-P_{23}=\frac{1}{H_1}\left(\frac{\partial^2 H_1}{\partial x_2\partial x_3}-\frac{1}{H_2}\frac{\partial H_2}{\partial x_3}\frac{\partial H_1}{\partial x_2}-\frac{1}{H_3}\frac{\partial H_3}{\partial x_2}\frac{\partial H_1}{\partial x_3}\right),$$

$$-P_{31}=\frac{1}{H_2}\left(\frac{\partial^2 H_2}{\partial x_3\partial x_1}-\frac{1}{H_3}\frac{\partial H_3}{\partial x_1}\frac{\partial H_2}{\partial x_3}-\frac{1}{H_1}\frac{\partial H_1}{\partial x_3}\frac{\partial H_2}{\partial x_1}\right),$$

$$-P_{12}=\frac{1}{H_3}\left(\frac{\partial^2 H_3}{\partial x_1\partial x_2}-\frac{1}{H_1}\frac{\partial H_1}{\partial x_2}\frac{\partial H_3}{\partial x_1}-\frac{1}{H_2}\frac{\partial H_2}{\partial x_1}\frac{\partial H_3}{\partial x_2}\right),$$

où

$$Q_1=\frac{\partial}{\partial x_2}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial x_3}\right)+\frac{1}{H_1^2}\frac{\partial H_2}{\partial x_1}\frac{\partial H_3}{\partial x_1},$$

$$Q_2=\frac{\partial}{\partial x_3}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial x_3}\right)+\frac{\partial}{\partial x_1}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial x_1}\right)+\frac{1}{H_2^2}\frac{\partial H_3}{\partial x_2}\frac{\partial H_1}{\partial x_2},$$

$$Q_3=\frac{\partial}{\partial x_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial x_2}\right)+\frac{1}{H_3^2}\frac{\partial H_1}{\partial x_3}\frac{\partial H_2}{\partial x_3}.$$

Je me borne maintenant à considérer le cas où

$$\text{(I)}\qquad P_{23}=0,\qquad P_{31}=0,\qquad P_{12}=0\text{ (}^1\text{)}.$$

D'après la loi de gravitation de M. Einstein nous avons ([2])

$$\text{(1)}\qquad \frac{\partial}{\partial x_1}\left(\frac{H_2H_3}{H_1}\frac{\partial f}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{H_3H_1}{H_2}\frac{\partial f}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{H_1H_2}{H_3}\frac{\partial f}{\partial x_3}\right)=0,$$

$$(2)_1\qquad \frac{\partial^2 f}{\partial x_1^2}=\frac{1}{H_1}\frac{\partial H_1}{\partial x_1}\frac{\partial f}{\partial x_1}-\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}\frac{\partial f}{\partial x_2}-\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\frac{\partial f}{\partial x_3}+fP_{11},$$

$$(2)_2\qquad \frac{\partial^2 f}{\partial x_2^2}=\frac{1}{H_2}\frac{\partial H_2}{\partial x_2}\frac{\partial f}{\partial x_2}-\frac{H_2}{H_3^2}\frac{\partial H_2}{\partial x_3}\frac{\partial f}{\partial x_3}-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial x_1}\frac{\partial f}{\partial x_1}+fP_{22},$$

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(1) Notre méthode est applicable au cas général, mais le résultat devient alors très compliqué.

(2) Voir H. Weyl, Raum, Zeit, Materie, 4e édition (1921), p. 219.

$$(2)_3 \qquad \frac{\partial^2 f}{\partial x_3^2}=\frac{1}{H_3}\frac{\partial H_3}{\partial x_3}\frac{\partial f}{\partial x_3}-\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial x_1}\frac{\partial f}{\partial x_1}-\frac{H_3}{H_2^2}\frac{\partial H_3}{\partial x_2}\frac{\partial f}{\partial x_2}+fP_{33},$$

$$(3)_1 \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2}=\frac{1}{H_1}\frac{\partial H_1}{\partial x_2}\frac{\partial f}{\partial x_1}+\frac{1}{H_2}\frac{\partial H_2}{\partial x_1}\frac{\partial f}{\partial x_2},$$

$$(3)_2 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_3}=\frac{1}{H_2}\frac{\partial H_2}{\partial x_3}\frac{\partial f}{\partial x_2}+\frac{1}{H_3}\frac{\partial H_3}{\partial x_2}\frac{\partial f}{\partial x_3},$$

$$(3)_3 \qquad \frac{\partial^2 f}{\partial x_3 \partial x_1}=\frac{1}{H_3}\frac{\partial H_3}{\partial x_1}\frac{\partial f}{\partial x_3}+\frac{1}{H_1}\frac{\partial H_1}{\partial x_3}\frac{\partial f}{\partial x_1}.$$

Nous allons chercher les conditions auxquelles doivent satisfaire les fonctions H_1, H_2, H_3, pour que la fonction f existe.

2. Tout d'abord, en tenant compte de $(2)_1$, $(2)_2$, $(2)_3$, l'équation (1) peut s'écrire

$$\begin{aligned}
&\frac{H_2H_3}{H_1}\Big(\frac{1}{H_1}\frac{\partial H_1}{\partial x_1}\frac{\partial f}{\partial x_1}-\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}\frac{\partial f}{\partial x_2}-\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\frac{\partial f}{\partial x_3}+fP_{11}\Big)\\
&\qquad\qquad+\Big(\frac{H_3}{H_1}\frac{\partial H_2}{\partial x_1}+\frac{H_2}{H_1}\frac{\partial H_3}{\partial x_1}-\frac{H_2H_3}{H_1^2}\frac{\partial H_1}{\partial x_1}\Big)\frac{\partial f}{\partial x_1}\\
&+\frac{H_3H_1}{H_2}\Big(\frac{1}{H_2}\frac{\partial H_2}{\partial x_2}\frac{\partial f}{\partial x_2}-\frac{H_2}{H_3^2}\frac{\partial H_2}{\partial x_3}\frac{\partial f}{\partial x_3}-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial x_1}\frac{\partial f}{\partial x_1}+fP_{22}\Big)\\
&\qquad\qquad+\Big(\frac{H_1}{H_2}\frac{\partial H_3}{\partial x_2}+\frac{H_3}{H_2}\frac{\partial H_1}{\partial x_2}-\frac{H_3H_1}{H_2^2}\frac{\partial H_2}{\partial x_2}\Big)\frac{\partial f}{\partial x_2}\\
&+\frac{H_1H_2}{H_3}\Big(\frac{1}{H_3}\frac{\partial H_3}{\partial x_3}\frac{\partial f}{\partial x_3}-\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial x_1}\frac{\partial f}{\partial x_1}-\frac{H_3}{H_2^2}\frac{\partial H_3}{\partial x_2}\frac{\partial f}{\partial x_2}+fP_{33}\Big)\\
&\qquad\qquad+\Big(\frac{H_2}{H_3}\frac{\partial H_1}{\partial x_3}+\frac{H_1}{H_3}\frac{\partial H_2}{\partial x_3}-\frac{H_1H_2}{H_3^2}\frac{\partial H_3}{\partial x_3}\Big)\frac{\partial f}{\partial x_3}\\
&=f\Big(\frac{H_2H_3}{H_1}P_{11}+\frac{H_3H_1}{H_2}P_{22}+\frac{H_1H_2}{H_3}P_{33}\Big)=0.
\end{aligned}$$

Nous avons donc

$$(\mathrm{II}) \qquad \frac{P_{11}}{H_1^2}+\frac{P_{22}}{H_2^2}+\frac{P_{33}}{H_3^2}=0,$$

$$H_1Q_1+H_2Q_2+H_3Q_3=0\,;$$

c'est la première condition nécessaire pour l'existence de la fonction f exprimant que la courbure de l'espace ayant l'élément linéaire $d\sigma$ est nulle. Si cette condition (II) est satisfaite, nous trouvons

$$P_{11}=\frac{H_1^2}{H_2H_3}Q_1, \qquad P_{22}=\frac{H_2^2}{H_3H_1}Q_2, \qquad P_{33}=\frac{H_3^2}{H_1H_2}Q_3.$$

Nous allons maintenant écrire quelques-unes des conditions d'intégrabilité des systèmes (2) et (3).

De l'équation $(2)_1$ nous avons

$$
\begin{aligned}
\frac{\partial}{\partial x_2}\left(\frac{\partial^2 f}{\partial x_1{}^2}\right) = {} & \frac{\partial \log H_1}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}\frac{\partial^2 f}{\partial x_2{}^2} - \frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\frac{\partial^2 f}{\partial x_2 \partial x_3} \\
& + \frac{\partial^2 \log H_1}{\partial x_1 \partial x_2}\frac{\partial f}{\partial x_1} - \frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}\right)\cdot\frac{\partial f}{\partial x_2} \\
& - \frac{\partial}{\partial x_2}\left(\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\right)\cdot\frac{\partial f}{\partial x_3} + P_{11}\frac{\partial f}{\partial x_2} + f\frac{\partial P_{11}}{\partial x_2} \\
= {} & \frac{\partial f}{\partial x_1}\left(\frac{\partial^2 \log H_1}{\partial x_1 \partial x_2} + \frac{\partial \log H_1}{\partial x_1}\frac{\partial \log H_1}{\partial x_2} + \frac{\partial \log H_1}{\partial x_2}\frac{\partial \log H_2}{\partial x_1}\right) \\
& + \frac{\partial f}{\partial x_2}\left[\frac{\partial \log H_1}{\partial x_1}\frac{\partial \log H_2}{\partial x_1} - \frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}\frac{\partial \log H_2}{\partial x_2}\right. \\
& \left. - \frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\frac{\partial \log H_2}{\partial x_3} - \frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}\right) + P_{11}\right] \\
& + \frac{\partial f}{\partial x_3}\left[\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_2}\frac{\partial \log H_2}{\partial x_3} - \frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\frac{\partial \log H_3}{\partial x_2}\right. \\
& \left. - \frac{\partial}{\partial x_2}\left(\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\right)\right] + f\left(\frac{\partial P_{11}}{\partial x_2} - \frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}P_{22}\right)
\end{aligned}
$$

d'après $(2)_2$, $(3)_1$ et $(3)_2$. De l'équation $(3)_1$ nous obtenons

$$
\begin{aligned}
\frac{\partial}{\partial x_1}\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right) = {} & \frac{\partial \log H_1}{\partial x_2}\frac{\partial^2 f}{\partial x_1{}^2} + \frac{\partial \log H_2}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 \log H_1}{\partial x_1 \partial x_2}\frac{\partial f}{\partial x_1} \\
& + \frac{\partial^2 \log H_2}{\partial x_1{}^2}\frac{\partial f}{\partial x_2} \\
= {} & \frac{\partial f}{\partial x_1}\left(\frac{\partial^2 \log H_1}{\partial x_1 \partial x_2} + \frac{\partial \log H_1}{\partial x_1}\frac{\partial \log H_1}{\partial x_2} + \frac{\partial \log H_2}{\partial x_1}\frac{\partial \log H_1}{\partial x_2}\right) \\
& + \frac{\partial f}{\partial x_2}\left[\frac{\partial^2 \log H_2}{\partial x_1{}^2} + \left(\frac{\partial \log H_2}{\partial x_1}\right)^2 - \frac{1}{H_2^2}\left(\frac{\partial H_1}{\partial x_2}\right)^2\right] \\
& - \frac{\partial f}{\partial x_3}\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\frac{\partial \log H_1}{\partial x_2} + fP_{11}\frac{\partial \log H_1}{\partial x_2}
\end{aligned}
$$

d'après $(2)_1$ et $(3)_1$.

Posons

$$
\frac{\partial}{\partial x_2}\left(\frac{\partial^2 f}{\partial x_1{}^2}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right) = \alpha\frac{\partial f}{\partial x_1} + \beta\frac{\partial f}{\partial x_2} + \gamma\frac{\partial f}{\partial x_3} + \delta f,
$$

et calculons les valeurs de α, β, γ et δ :

$$\alpha = 0,$$

$$\beta = P_{11} - \frac{H_1}{H_2}\left(\frac{1}{H_1}\frac{\partial^2 H_2}{\partial x_1^2} - \frac{1}{H_1^2}\frac{\partial H_1}{\partial x_1}\frac{\partial H_2}{\partial x_2} + \frac{1}{H_2}\frac{\partial^2 H_2}{\partial x_2^2} - \frac{1}{H_2^2}\frac{\partial H_2}{\partial x_2}\frac{\partial H_1}{\partial x_2} + \frac{1}{H_3}\frac{\partial H_1}{\partial x_3}\frac{\partial H_2}{\partial x_3}\right)$$

$$= P_{11} - \frac{H_1}{H_2} Q_3$$

$$= H_1^2\left(\frac{P_{11}}{H_1^2} - \frac{P_{33}}{H_3^2}\right),$$

$$\gamma = -\frac{H_1}{H_3^2}\left(\frac{\partial^2 H_1}{\partial x_2 \partial x_3} - \frac{1}{H_2}\frac{\partial H_1}{\partial x_2}\frac{\partial H_2}{\partial x_3} - \frac{1}{H_3}\frac{\partial H_1}{\partial x_3}\frac{\partial H_3}{\partial x_2}\right)$$

$$= \frac{H_1^2}{H_3^2} P_{23} = 0,$$

$$\delta = \frac{\partial P_{11}}{\partial x_2} - H_1 \frac{\partial H_1}{\partial x_2}\left(\frac{P_{11}}{H_1^2} + \frac{P_{22}}{H_2^2}\right)$$

$$= \frac{\partial P_{11}}{\partial x_2} + \frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_2} P_{33}.$$

Par conséquent, pour que $\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial^2 f}{\partial x_1^2}\right)$ soit égale à $\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\right)$, il faut que

$$H_1\left(\frac{P_{33}}{H_3^2} - \frac{P_{11}}{H_1^2}\right)\frac{\partial f}{\partial x_2} - \left(\frac{1}{H_1}\frac{\partial P_{11}}{\partial x_2} + \frac{P_{33}}{H_3^2}\frac{\partial H_1}{\partial x_2}\right) f = 0.$$

De même, si nous posons

$$\left\{\begin{aligned}
L_{21} &= H_3\left(\frac{P_{22}}{H_2^2} - \frac{P_{33}}{H_3^2}\right),\\
L_{31} &= H_2\left(\frac{P_{33}}{H_3^2} - \frac{P_{22}}{H_2^2}\right),\\
L_{32} &= H_1\left(\frac{P_{33}}{H_3^2} - \frac{P_{11}}{H_1^2}\right),\\
L_{12} &= H_3\left(\frac{P_{11}}{H_1^2} - \frac{P_{33}}{H_3^2}\right),\\
L_{13} &= H_2\left(\frac{P_{11}}{H_1^2} - \frac{P_{22}}{H_2^2}\right),\\
L_{23} &= H_1\left(\frac{P_{22}}{H_2^2} - \frac{P_{11}}{H_1^2}\right),
\end{aligned}\right.
\qquad
\left\{\begin{aligned}
M_{21} &= \frac{1}{H_3}\frac{\partial P_{33}}{\partial x_1} + \frac{P_{22}}{H_2^2}\frac{\partial H_3}{\partial x_1},\\
M_{31} &= \frac{1}{H_2}\frac{\partial P_{22}}{\partial x_1} + \frac{P_{33}}{H^2}\frac{\partial H_2}{\partial x_1},\\
M_{32} &= \frac{1}{H_1}\frac{\partial P_{11}}{\partial x_2} + \frac{P_{33}}{H_3^2}\frac{\partial H_1}{\partial x_2},\\
M_{12} &= \frac{1}{H_3}\frac{\partial P_{33}}{\partial x_2} + \frac{P_{11}}{H_1^2}\frac{\partial H_3}{\partial x_2},\\
M_{13} &= \frac{1}{H_2}\frac{\partial P_{22}}{\partial x_3} + \frac{P_{11}}{H_1^2}\frac{\partial H_2}{\partial x_3},\\
M_{23} &= \frac{1}{H_1}\frac{\partial P_{11}}{\partial x_3} + \frac{P_{22}}{H_2^2}\frac{\partial H_1}{\partial x_3},
\end{aligned}\right.$$

nous obtenons les six équations suivantes :

$$(4)\quad \begin{cases} L_{21}\dfrac{\partial f}{\partial x_1}=M_{21}f, & L_{32}\dfrac{\partial f}{\partial x_2}=M_{32}f, & L_{13}\dfrac{\partial f}{\partial x_3}=M_{13}f, \\ L_{31}\dfrac{\partial f}{\partial x_1}=M_{31}f, & L_{12}\dfrac{\partial f}{\partial x_2}=M_{12}f, & L_{23}\dfrac{\partial f}{\partial x_3}=M_{23}f. \end{cases}$$

Nous avons donc *la deuxième condition nécessaire* pour l'existence de la fonction f :

$$\text{(III)}\quad \frac{M_{21}}{L_{21}}=\frac{M_{31}}{L_{31}}\left(=A_1\right),\quad \frac{M_{12}}{L_{12}}=\frac{M_{32}}{L_{32}}\left(=A_2\right),\quad \frac{M_{23}}{L_{23}}=\frac{M_{13}}{L_{13}}(=A_3);$$

puis *la troisième condition nécessaire :*

$$\text{(IV)}\qquad \frac{\partial A_2}{\partial x_3}=\frac{\partial A_3}{\partial x_2},\quad \frac{\partial A_3}{\partial x_1}=\frac{\partial A_1}{\partial x_3},\quad \frac{\partial A_1}{\partial x_2}=\frac{\partial A_2}{\partial x_1}.$$

Si toutes les conditions (II), (III) et (IV) sont satisfaites, en intégrant (4), *nous trouvons*

$$(5)\qquad f=c\,e^{\int A_1 dx_1+A_2 dx_2+A_3 dx_3},$$

c étant une constante.

3. Mais il ne faut pas croire que les conditions (II), (III), (IV) soient suffisantes pour l'existence de la fonction f qui satisfait aux équations (1), (2) et (3). En effet, si nous prenons par exemple

$$H_1=x_2^2-x_3^2,\qquad H_2=1,\qquad H_3=1,$$

nous avons

$$P_{11}=0,\qquad P_{22}=-\frac{2}{x_2^2-x_3^2},\qquad P_{33}=\frac{2}{x_2^2-x_3^2};$$

$$P_{23}=0,\qquad P_{31}=0,\qquad P_{12}=0;$$

$$L_{21}=-L_{31}=-\frac{4}{x_2^2-x_3^2},\quad L_{32}=-L_{23}=2,\quad L_{12}=-L_{13}=-\frac{2}{x_2^2-x_3^2},$$

$$M_{21}=0,\qquad M_{32}=\frac{4x_2}{x_2^2-x_3^2},\qquad M_{23}=\frac{4x_3}{x_2^2-x_3^2},$$

$$M_{31}=0,\qquad M_{12}=-\frac{4x_2}{(x_2^2-x_3^2)^2},\quad M_{13}=-\frac{4x_3}{(x_2^2-x_3^2)^2};$$

$$A_1=0,\qquad A_2=\frac{2x_2}{x_2^2-x_3^2},\qquad A_3=-\frac{2x_3}{x_2^2-x^2}.$$

Ainsi toutes les conditions (I), (II), (III), (IV) sont satisfaites, mais la fonction

$$f=c\,e^{\int\frac{2x_2dx_2-2x_3dx_3}{x_2^2-x_3^2}}=c(x_2^2-x_3^2)$$

donnée par (5) ne satisfait pas à l'équation (1); car

$$\frac{\partial}{\partial x_2}\left[x_2(x_2^2-x_3^2)\right]-\frac{\partial}{\partial x_3}\left[x_3(x_2^2-x_3^2)\right]=4(x_2^2+x_3^2)\neq 0.$$

Par conséquent, *il faut examiner si la fonction f donnée par (5) satisfait aux équations (1), (2) et (3).*

4. Pour montrer quelques applications de la méthode que nous venons d'indiquer, nous allons commencer avec la forme de Schwarzschild-Eddington[1].

Prenons la forme

$$d\sigma^2=e^{\lambda(x_1)}dx_1^2+x_1^2dx_2^2+x_1^2\sin^2 x_2\,dx_3^2.$$

Nous avons

$$P_{11}=\frac{\lambda'}{x_1},\quad P_{22}=e^{-\lambda}\left(e^{\lambda}-1+\frac{1}{2}x_1\lambda'\right),\quad P_{33}=e^{-\lambda}\left(e^{\lambda}-1+\frac{1}{2}x_1\lambda'\right)\sin^2 x_2,$$

$$P_{12}=0,\qquad P_{23}=0,\qquad P_{31}=0,$$

où l'accent désigne la dérivation par rapport à x_1. Donc la condition (I) est satisfaite, et (II) devient

$$x_1\lambda'+e^{\lambda}-1=0,$$

d'où

$$e^{\lambda}=\left(1-\frac{2m}{x_1}\right)^{-1}\qquad (m \text{ étant const.}),$$

et par suite

$$P_{11}=-\frac{2m}{x_1^3\left(1-\frac{2m}{x_1}\right)},\quad P_{22}=\frac{m}{x_1},\quad P_{33}=\frac{m}{x_1}\sin^2 x_2.$$

Puisque l'on a

$$L_{21}=L_{31}=0,\quad L_{32}\neq 0,\quad L_{12}\neq 0,\quad L_{13}\neq 0,\quad L_{23}\neq 0,$$
$$M_{21}=M_{31}=M_{32}=M_{12}=M_{13}=M_{23}=0,$$

en vertu de (4), f *doit* être une fonction de la seule variable x_1. Dans ce cas l'équation $(2)_2$ prend la forme

$$\frac{df}{dx_1}=-\frac{f}{2}\frac{d\lambda}{dx_1},$$

d'où

$$f=ce^{-\frac{\lambda}{2}}=c\left(1-\frac{2m}{x_1}\right)^{\frac{1}{2}},$$

(1) Il y a plusieurs méthodes de l'intégration des équations de gravitation pour ce cas. Mais je crois que la méthode que je vais indiquer est une des plus simples. Pour une autre méthode tout à fait différente, voir K. Ogura, "sur la théorie de gravitation dans l'espace à deux dimensions," Comptes Rendus, Paris, t. 173 (1921), p. 909.

c étant une constante. Nous avons ainsi trouvé *la forme de* Schwarzschild-Eddington:

$$(6)\quad ds^2 = c^2\left(1-\frac{2m}{x_1}\right)dt^2 - \frac{dx_1^2}{1-\frac{2m}{x_1}} - x_1^2 dx_2^2 - x_1^2 \sin^2 x_2 dx_3^2 \text{ (}^1\text{)}.$$

5. 1°. Considérons maintenant la forme

$$(7)\qquad d\sigma^2 = H_1^2(x_1, x_2)\, dx_1^2 + H_2^2(x_1, x_2)\, dx_2^2 + dx_3^2.$$

Si l'on désigne par K

$$-\frac{1}{H_1H_2}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial x_2}\right)\right]$$

la courbure totale de la surface ayant l'élément linéaire

$$(d\sigma)^2_{x_3=\text{const.}} = H_1^2(x_1, x_2)\, dx_1^2 + H_2^2(x_1, x_2)\, dx_2^2,$$

les tenseurs P_{ik} peuvent s'écrire

$$P_{11} = H_1^2 K, \qquad P_{22} = H_2^2 K, \qquad P_{33} = 0,$$
$$P_{23} = 0, \qquad P_{31} = 0, \qquad P_{12} = 0.$$

Ainsi la condition (I) est satisfaite, et (II) nous donne

$$K = 0,$$

d'où

$$P_{11} = 0, \qquad P_{22} = 0.$$

Par conséquent, *l'espace vide ayant l'élément linéaire* (7) *est euclidien, et les surfaces* $x_3 = \text{const.}$ *peuvent être considérées comme des plans euclidiens.*

2°. Prenons ensuite la forme

$$(8)_1\qquad d\sigma^2 = dx_1^2 + \cos^2 x_1\, dx_2^2 + \cos^2 x_1 \cos^2 x_2\, dx_3^2.$$

Puisque

$$P_{11} = 2, \qquad P_{22} = 2\cos^2 x_1, \qquad P_{33} = 2\cos^2 x_1 \cos^2 x_2,$$
$$P_{23} = 0, \qquad P_{31} = 0, \qquad P_{12} = 0,$$

la condition (I) est satisfaite, mais pas (II). Donc *il n'existe pas d'espace sphérique vide*(²).

(¹) Il ne faut pas croire que la surface $x_2 = \frac{\pi}{2}$ dans l'espace avec l'élément linéaire $d\sigma$ soit un plan euclidien. En fait, la courbure totale K de cette surface est égale à $-\frac{m}{x_1^3}$.

(²) Si nous considérons la loi modifiée $R_{\mu\nu} = \lambda g_{\mu\nu}$ au lieu de $R_{\mu\nu} = 0$, cet espace devient très important.

De même, on peut démontrer *qu'il n'existe pas d'espace hyperbolique vide* ayant l'élément linéaire

$$(8)_2 \qquad d\sigma^2 = dx_1^2 + \mathrm{ch}^2 x_1\, dx_2^2 + \mathrm{ch}^2 x_1\, \mathrm{ch}^2 x_2\, dx_3^2.$$

Quelques généralisations d'un théorème de Liouville au champ de gravitation.

6. Dans la géométrie euclidienne à $n(n \geqq 3)$ dimensions nous avons le théorème suivant (dû à J. Liouville pour le cas où $n=3$): Pour que l'élément linéaire de l'espace puisse être représenté par la forme

$$d\sigma^2 = \lambda^2(x_1, x_2, \ldots, x_n)(dx_1^2 + dx_2^2 + \ldots + dx_n^2), \quad n \geqq 3,$$

il faut et il suffit que

$$\lambda = \text{const.}, \text{ ou } \quad \lambda = \frac{c}{(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2},$$

$c, a_1, \ldots, a_n$ étant constantes.

Et l'on sait que pour démontrer ce théorème analytiquement il suffit de considérer la forme([1])

$$\lambda = \frac{1}{X_1(x_1) + \ldots + X_n(x_n)},$$

où $X_i(x_i)$ désigne une fonction de la seule variable x_i.

Considérons maintenant un champ *statique* de gravitation dans un espace *vide*. Quand l'intervalle élémentaire prend la forme

$$ds^2 = f^2 dt^2 - d\sigma^2 = f^2 dt^2 - \lambda^2(x_1, x_2, x_3)(dx_1^2 + dx_2^2 + dx_3^2),$$

peut-on mettre la fonction λ toujours sous la forme

$$\lambda = \text{const.}, \text{ ou } \lambda = \frac{c}{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} ?$$

Non! En effet, si nous opérons dans (6) la substitution

$$x_1 = r\left(1 + \frac{m}{2r}\right)^2, \; x = r \sin x_2 \cos x_3, \; y = r \sin x_2 \sin x_3, \; z = r \cos x_2,$$

nous obtenons

$$(6)' \quad ds^2 = c^2 \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2} dt^2 - \left(1 + \frac{m}{2r}\right)^4 (dx^2 + dy^2 + dz^2)$$
$$(r = \sqrt{x^2 + y^2 + z^2}),$$

([1]) Voir L. Bianchi, Vorlesungen über Differentialgeometrie, 2.e édition (1910), p. 636.

ce qui tend vers

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

pour l'infini.

Modifions un peu notre problème et *cherchons tous les espace-temps dont les intervalles élémentaires prennent la forme*

$$(9) \qquad ds^2 = f^2 dt^2 - d\sigma^2 = f^2 dt^2 - \frac{dx_1^2 + dx_2^2 + dx_3^2}{[X_1(x_1) + X_2(x_2) + X_3(x_3)]^2}.$$

7. Pour cette expression (9) nous avons

$$H_1 = H_2 = H_3 = \frac{1}{X_1(x_1) + X_2(x_2) + X_3(x_3)},$$

$$(10) \quad \begin{cases} P_{11} = \dfrac{(X_1 + X_2 + X_3)(2X_1'' + X_2'' + X_3'') - 2(X'^2_1 + X'^2_2 + X'^2_3)}{(X_1 + X_2 + X_3)^2}, \\ P_{22} = \dfrac{(X_1 + X_2 + X_3)(X_1'' + 2X_2'' + X_3'') - 2(X'^2_1 + X'^2_2 + X'^2_3)}{(X_1 + X_2 + X_3)^2}, \\ P_{33} = \dfrac{(X_1 + X_2 + X_3)(X_1'' + X_2'' + 2X_3'') - 2(X'^2_1 + X'^2_2 + X'^2_3)}{(X_1 + X_2 + X_3)^2}, \end{cases}$$

$$P_{23} = 0, \qquad P_{31} = 0, \qquad P_{12} = 0,$$

où X_i' désigne la dérivée de X_i par rapport à x_i. La condition (I) est donc satisfaite, et (II) devient

$$(\text{II}) \qquad 2(X_1 + X_2 + X_3)(X_1'' + X_2'' + X_3'') - 3(X'^2_1 + X'^2_2 + X'^2_3) = 0.$$

Par définition, on a

$$L_{13} = \frac{3(X'^2_1 + X'^2_2 + X'^2_3) - (X_1 + X_2 + X_3)(X_1'' + 3X_2'' + 2X_3'')}{X_1 + X_2 + X_3}$$

$$\begin{aligned} &= X_1'' - X_2'', & L_{23} &= -(X_1'' - X_2''), \\ L_{21} &= X_2'' - X_3'', & L_{31} &= -(X_2'' - X_3''), \\ L_{32} &= X_3'' - X_1'', & L_{12} &= -(X_3'' - X_1''); \end{aligned}$$

$$M_{13} = M_{23}$$

$$= \frac{(X_1 + X_2 + X_3)^2 X_3''' - 3X_3'(X_1 + X_2 + X_3)(X_1'' + X_2'' + 2X_3'') + 6X_3'(X'^2_1 + X'^2_2 + X'^2_3)}{(X_1 + X_2 + X_3)^2}$$

$$= X_3''' + \frac{X_3'(X_1'' + X_2'' - 2X_3'')}{X_1 + X_2 + X_3},$$

$$M_{21} = M_{31} = X_1''' + \frac{X_1'(X_2'' + X_3'' - 2X_1'')}{X_1 + X_2 + X_3},$$

$$M_{32} = M_{12} = X_2''' + \frac{X_2''(X_3'' + X_1'' - 2X_2'')}{X_1 + X_2 + X_3}.$$

8. 1° Tout d'abord nous allons démontrer que *toutes les quantités*

M_{ik} $(i, k=1, 2, 3\,;\ i\neq k)$ *sont nulles.* Car, si $M_{21}(=M_{31})$, par exemple, n'était pas nulle, la condition (III) donnerait

$$X_2''-X_3''=0\,;$$

puis (4) devient $f=0$, ce qui n'est pas possible.

2°. Ensuite, *il y a, au moins, deux quantités qui sont égales entre les trois* X_1'', X_2'' et X_3''. Car, si toutes les trois X_1'', X_2'', X_3'' étaient différentes, on aurait

$$L_{21}\neq 0, \qquad L_{32}\neq 0. \qquad L_{13}\neq 0.$$

Mais, puisque

$$M_{21}=0, \qquad M_{32}=0, \qquad M_{13}=0,$$

on déduit de (4), que f est une constante. Il suit de $(2)_1$, $(2)_2$, $(2)_3$ que

$$P_{11}=0, \qquad P_{22}=0, \qquad P_{33}=0\,;$$

puis les équations (10) nous donnent

$$X_1''=X_2''=X_3'',$$

ce qui contredit notre supposition.

9. 1°. Nous allons maintenant considérer le cas où

$$X_1''=X_2''=X_3''\,(=k).$$

Puisque k doit être une constante, (11) devient

$$(X_1'^2+X_2'^2+X_3'^2)-2k(X_1+X_2+X_3)=0\,;$$

puis (10) devient

$$P_{11}=0, \qquad P_{22}=0, \qquad P_{33}=0.$$

C'est justement le cas du théorème de Liouville dans la géométrie euclidienne[1], et nous avons donc

$$d\sigma^2=c^2(dx_1^2+dx_2^2+dx_3^2) \text{ ou } d\sigma^2=\frac{c^2(dx_1^2+dx_2^2+dx_3^2)}{[(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2]^2}.$$

2° Considérons le second cas où il y a deux quantités qui sont égales entre les trois X_1'', X_2'', X_3''. Prenons, par exemple, le cas où

$$X_2''=X_3'', \qquad X_3''\neq X_1'', \qquad X_1''\neq X_2''.$$

Puisque l'on a

$$L_{21}=L_{31}=0, \quad L_{32}\neq 0, \quad L_{12}\neq 0, \quad L_{13}\neq 0, \quad L_{23}\neq 0,$$

les équations (4) se réduissent aux deux équations

(1) Voir Bianchi, p. 637.

$$\frac{\partial f}{\partial x_2}=0\,, \qquad \frac{\partial f}{\partial x_3}=0\,;$$

par conséquent, f est une fonction de la seule variable x_1.

Si nous posons

$$X_2''=X_3''=k,$$

k doit être une constante; nous avons donc $X_2'''=0$, $X_3'''=0$, et les équations

$$M_{32}=0, \qquad M_{13}=0$$

deviennent

$$X_2'(X_1''-k)=0, \qquad X_3'(X_1''-k)=0,$$

d'où

$$X_2'=0, \qquad X_3'=0\,;$$

car $X_1''\neq k=X_2''$. Nous avons donc

$$k=0, \quad X_2=a_2, \quad X_3=a_3, \quad (a_2,\ a_3 \text{ étant const.});$$

puis (11) devient

$$2(X_1+a_2+a_3)X_1''-2X_1'^2=0,$$

d'où nous trouvons

$$X_1+a_2+a_3=\frac{1}{(ax_1+b)^2}\,,$$

a, b étant constantes, et

$$d\sigma^2=(ax_1+b)^4\,(dx_1^2+dx_2^2+dx_3^2).$$

Puisque (10) donne

$$P_{11}=\frac{4a^2}{(ax_1+b)^2}\,, \qquad P_{22}=P_{33}=-\frac{2a^2}{(ax_1+b)^2}\,,$$

l'équation $(2)_2$ devient

$$\frac{df}{dx_1}+\frac{a}{ax_1+b}f=0\,,$$

d'où

$$f=\frac{c}{ax_1+b}\,, \qquad (c \text{ étant const.}).$$

Ainsi nous obtenons la forme

$$ds^2=\frac{c^2}{(ax_1+b)^2}\,dt^2-(ax_1+b)^4\,(dx_1^2+dx_2^2+dx_3^2),$$

et il est facile de prouver que cette forme satisfait aux équations (1), (2), (3).

De la même manière, on peut traiter les cas où

$$X_3'' = X_1'', \quad X_1'' \neq X_2'', \quad X_2'' \neq X_3''$$

et

$$X_1'' = X_2'', \quad X_2'' \neq X_3'', \quad X_3'' \neq X_1''.$$

Nous avons ainsi établi le théorème suivant:

Pour que l'intervalle élémentaire d'un champ statique de gravitation dans un espace vide puisse être représenté par la forme

$$ds^2 = f^2 dt^2 - \frac{dx_1^2 + dx_2^2 + dx_3^2}{(X_1 + X_2 + X_3)^2} = f^2 dt^2 - \lambda^2 (dx_1^2 + dx_2^2 + dx_3^2),$$

il faut et il suffit que

$$\lambda = const. \quad ou \quad \lambda = \frac{c}{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} \quad (cas\ euclidien),$$

ou bien

$$ds^2 = \frac{c}{(ax_i + b)} dt^2 - (ax_i + b)^4 (dx_1^2 + dx_2^2 + dx_3^2) \quad (cas\ noneuclidien),$$

$$(i = 1, 2. 3),$$

a, b, c étant constantes.

10. Nous allons étudier en détail l'espace-temps défini par

$$(12) \qquad ds^2 = \frac{c^2}{(ax_1 + b)^2} dt^2 - (ax_1 + b)^4 (dx_1^2 + dx_2^2 + dx_3^2).$$

Pour cette forme les symboles de Christoffel deviennent

$$\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 3 \end{Bmatrix} = \frac{2a}{ax_1 + b}, \qquad \begin{Bmatrix} 14 \\ 4 \end{Bmatrix} = -\frac{a}{ax_1 + b},$$

$$\begin{Bmatrix} 22 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 33 \\ 1 \end{Bmatrix} = -\frac{2a}{ax_1 + b}, \qquad \begin{Bmatrix} 44 \\ 1 \end{Bmatrix} = -\frac{ac^2}{(ax_1 + b)^7}, \quad (x_4 = t),$$

et les autres sont nuls. Puisque le tenseur de Riemann-Christoffel

$$R^{\epsilon}_{\mu\nu\sigma} = \sum_{\alpha=1}^{4} \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha\sigma \\ \varepsilon \end{Bmatrix} - \sum_{\alpha=1}^{4} \begin{Bmatrix} \mu\sigma \\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha\nu \\ \varepsilon \end{Bmatrix} + \frac{\partial}{\partial x_\sigma} \begin{Bmatrix} \mu\nu \\ \varepsilon \end{Bmatrix} - \frac{\partial}{\partial x_\nu} \begin{Bmatrix} \mu\sigma \\ \varepsilon \end{Bmatrix} \quad (x_4 = t)$$

prend la forme

$$R^{\epsilon}_{\mu\nu\sigma} = \sum_m \frac{k_m}{(ax_1 + b)^m} \qquad m > 1,$$

(k_m étant constante); nous trouvons

$$\lim_{x_1 \to \infty} R^{\epsilon}_{\mu\nu\sigma} \to 0.$$

Par conséquent, notre espace-temps a une (seule) singularité pour

$$x_1 = -\frac{b}{a};$$

et il devient euclidien lorsque x_1 tend vers l'infini. Mais, pour $x_1 = \infty$, li ne prend pas la forme

$$ds^2 = c^2 dt^2 - (dx_1^2 + dx_2^2 + dx_3^2).$$

Ainsi nous avons le théorème suivant :

Pour que l'intervalle élémentaire d'un champ statique de gravitation dans un espace vide puisse être représenté par la forme

$$ds^2 = f^2 dt^2 - \frac{dx_1^2 + dx_2^2 + dx_3^2}{(X_1 + X_2 + X_3)^2}$$

avec la condition (¹)

$$f \to c, \qquad (X_1 + X_2 + X_3)^2 \to 1 \qquad \textit{pour l'infini},$$

il faut et il suffit que l'espace-temps soit euclidien.

Posons maintenant

$$(13) \qquad x_1 = \frac{(3ax)^{\frac{1}{3}} - b}{a}, \quad x_2 = y(3ax)^{-\frac{2}{3}}, \quad x_3 = z(3ax)^{-\frac{2}{3}},$$

$$t = \tau(3ax)^{\frac{1}{3}}.$$

En utilisant ces nouvelles variables, nous pouvons mettre (12) sous la forme

$$(14) \quad ds^2 = c^2\left(d\tau + \frac{1}{3}\frac{\tau}{x}dx\right)^2 - dx^2 - \left(dy - \frac{2}{3}\frac{y}{x}dx\right)^2 - \left(dz - \frac{2}{3}\frac{z}{x}dx\right)^2,$$

dont le déterminant des coefficients est

$$g = \begin{vmatrix} -1 - \frac{4}{9}\frac{y^2}{x^2} - \frac{4}{9}\frac{z^2}{x^2} + \frac{c^2}{9}\frac{\tau^2}{x^2} & \frac{2}{3}\frac{y}{x} & \frac{2}{3}\frac{z}{x} & \frac{c^2}{3}\frac{\tau}{x} \\ \frac{2}{3}\frac{y}{x} & -1 & 0 & 0 \\ \frac{2}{3}\frac{z}{x} & 0 & -1 & 0 \\ \frac{c^2}{3}\frac{\tau}{x} & 0 & 0 & c^2 \end{vmatrix}$$

$$= -c^2.$$

Cet espace-temps *n'est pas statique*, mais *bien coordonné*. (Les coordonnées forment un "berechtige Bezugssystem" au sens de M. Kottler(²).)

Le champ a une (seule) singularité pour $x = 0$; et il tend vers la forme

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(¹) Voir M. v. Laue, Relativitätstheorie, t. 2 (1921), p. 179. Il faut remarquer que *le théorème analogue n'est pas valable pour le cas plus général :*

$$ds^2 = f^2 dt^2 - \lambda^2(x_1, x_2, x_3)(dx_1^2 + dx_2^2 + dx_3^2).$$

Voir la forme (6)′ du §6.

(²) F. Kottler, Annalen der Physik, t. 56 (1918), p. 1.

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2$$

lorsque x tend vers l'infini. D'après M. Hilbert, la condition pour la réalité de cet espace-temps est donnée par

$$c^2\tau^2 < qx^2.$$

11. Cherchons maintenant une autre généralisation du théorème de Liouville.

Considérons un champ de gravitation dans un espace *vide*, et supposons que l'intervalle élémentaire de ce champ puisse être représenté par

$$ds^2 = f^2(t, x_1, x_2, x_3)\,(dt^2 - dx_1^2 - dx_2^2 - dx_3^2). \tag{15}$$

D'après la loi de gravitation, nous avons

$$(16)_1 \quad 3\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} - \frac{\partial^2 f}{\partial t^2} + \frac{1}{f}\left[-3\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 - \left(\frac{\partial f}{\partial t}\right)^2\right] = 0,$$

$$(16)_2 \quad \frac{\partial^2 f}{\partial x_1^2} + 3\frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} - \frac{\partial^2 f}{\partial t^2} + \frac{1}{f}\left[\left(\frac{\partial f}{\partial x_1}\right)^2 - 3\left(\frac{\partial f}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2 - \left(\frac{\partial f}{\partial t}\right)^2\right] = 0,$$

$$(16)_3 \quad \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + 3\frac{\partial^2 f}{\partial x_3^2} - \frac{\partial^2 f}{\partial t^2} + \frac{1}{f}\left[\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 - 3\left(\frac{\partial f}{\partial x_3}\right)^2 - \left(\frac{\partial f}{\partial t}\right)^2\right] = 0,$$

$$(16)_4 \quad \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} - 3\frac{\partial^2 f}{\partial t^2} + \frac{1}{f}\left[\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 + 3\left(\frac{\partial f}{\partial t}\right)^2\right] = 0,$$

$$(16)_5 \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{2}{f}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} = 0, \qquad (16)_8 \quad \frac{\partial^2 f}{\partial x_1 \partial t} - \frac{2}{f}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial t} = 0,$$

$$(16)_6 \quad \frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{2}{f}\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial x_3} = 0, \qquad (16)_9 \quad \frac{\partial^2 f}{\partial x_2 \partial t} - \frac{2}{f}\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial t} = 0,$$

$$(16)_7 \quad \frac{\partial^2 f}{\partial x_3 \partial x_1} - \frac{2}{f}\frac{\partial f}{\partial x_3}\frac{\partial f}{\partial x_1} = 0, \qquad (16)_{10} \quad \frac{\partial^2 f}{\partial x_3 \partial t} - \frac{2}{f}\frac{\partial f}{\partial x_3}\frac{\partial f}{\partial t} = 0.$$

Les équations $(16)_5 - (16)_{10}$ nous donnent

$$f = \frac{1}{X_1(x_1) + X_2(x_2) + X_3(x_3) + T(t)}.$$

En substituant cette forme de f dans $(16)_1-(16)_4$, on trouve

$$\frac{3(X_1'^2+X_2'^2+X_3'^2-T'^2)}{X_1+X_2+X_3+T}=3X_1''+X_2''+X_3''-T''$$
$$=X_1''+3X_2''+X_3''-T''$$
$$=X_1''+X_2''+3X_3''-T''$$
$$=X_1''+X_2''+X_3''-3T'',$$

d'où

$$(17)\quad X_1''=X_2''=X_3''=-T''$$
$$=\frac{X_1'^2+X_2'^2+X_3'^2-T'^2}{2(X_1+X_2+X_3+T)}=k,$$

k ne pouvant être qu'une constante.

12. 1°. Examinons d'abord le cas où $k\neq 0$.

En désignant par $a_1,\ldots,a_4$, $b_1,\ldots,b_4$ des constantes, nous déduisons, de (17),

$$X_i=\frac{k}{2}x_i^2+a_ix_i+b_i\quad (i=1,2,3),\qquad T=-\frac{k}{2}t^2+a_4t+b_4,$$

$$(kx_1+a_1)^2+\ldots\ldots-(-kt+a_4)^2=2k\Big[\Big(\frac{k}{2}x_1^2+a_1x_1+b_1\Big)+\ldots\ldots$$
$$+\Big(-\frac{k}{2}t^2+a_4t+b_4\Big)\Big],$$

d'où

$$a_1^2+a_2^2+a_3^2-a_4^2=2k(b_1+b_2+b_3+b_4).$$

Il suit que

$$X_1+X_2+X_3+T=\frac{k}{2}\Big[\Big(x_1+\frac{a_1}{k}\Big)^2+\Big(x_2+\frac{a_2}{k}\Big)^2+\Big(x_3+\frac{a_3}{k}\Big)^2-\Big(t-\frac{a_4}{k}\Big)^2\Big]$$
$$+\frac{1}{2k}\Big[2k(b_1+b_2+b_3+b_4)-(a_1^2+a_2^2+a_3^2-a_4^2)\Big]$$
$$=\frac{k}{2}\Big[\Big(x_1+\frac{a_1}{k}\Big)^2+\Big(x_2+\frac{a_2}{k}\Big)^2+\Big(x_3+\frac{a_3}{k}\Big)^2-\Big(t-\frac{a_4}{k}\Big)^2\Big].$$

Par conséquent, nous obtenons

$$(18)\quad ds^2=\frac{c^2(dt^2-dx_1^2-dx_2^2-dx_3^2)}{[(t-c_4)^2-(x_1-c_1)^2-(x_2-c_2)^2-(x_3-c_3)^2]^2},$$

$c, c_1,\ldots, c_4$ étant constantes.

2°. Examinons ensuite le cas où $k=0$.

Si $a_1=a_2=a_3=a_4=0$, toutes les quantités X_1, X_2, X_3, T sont constantes, et nous avons

$$(19) \qquad ds^2 = c^2(dt^2 - dx_1^2 - dx_2^2 - dx_3^2),$$

c étant constante.

Si toutes les constantes a_1, a_2, a_3, a_4 ne sont pas nulles, nous trouvons, de (17),

$$X_i = a_i x_i + b_i \quad (i=1, 2, 3), \qquad T = a_4 t + b_4,$$
$$a_1^2 + a_2^2 + a_3^2 - a_4^2 = 0;$$

puis

$$(20) \qquad ds^2 = \frac{dt^2 - dx_1^2 - dx_2^2 - dx_3^2}{(k_4 t - k_1 x_1 - k_2 x_2 - k_3 x_3 - k_5)^2},$$

$k_1, k_2, \ldots, k_5$ désignant des constantes avec la condition

$$k_1^2 + k_2^2 + k_3^2 = k_4^2.$$

Mais, on sait qu'on peut, à l'aide des transformations bien connues([1]), réduire ds (de (18) et de (20)) à la forme

$$ds^2 = d\bar{t}^2 - d\bar{x}_1^2 - d\bar{x}_2^2 - d\bar{x}_3^2.$$

Nous avons donc établi le théorème suivant :

Pour que l'intervalle élémentaire d'un champ de gravitation d'un espace vide puisse être représenté par la forme

$$ds^2 = f^2(t, x_1, x_2, x_3)\,(dt^2 - dx_1^2 - dx_2^2 - dx_3^2),$$

il faut et il suffit que

$$f = \textit{constante},$$

ou

$$f = \frac{c}{(t-a_4)^2 - (x_1 - a_1)^2 - (x_2 - a_2)^2 - (x_3 - a_3)^2},$$

ou

$$= \frac{1}{b_4 t - b_1 x_1 - b_2 x_2 - b_3 x_3 - b_4} \qquad (b_1^2 + b_2^2 + b_3^2 = b_4^2),$$

a_i, b_i, c étant constantes. Et chacun de ces espace-temps est euclidien.

Ainsi, *la détermination de la forme*

$$ds^2 = f^2(t, x_1, x_2, x_3)\,(dt^2 - dx_1^2 - dx_2^2 - dx_3^2)$$

pour le champ de gravitation de l'espace vide est équivalente de la détermination de la transformation :

$$d\bar{t}^2 - d\bar{x}_1^2 - d\bar{x}_2^2 - d\bar{x}_3^2 = f^2(t, x_1, x_2, x_3)\,(dt^2 - dx_1^2 - dx_2^2 - dx_3^2).$$

Dans le cas particulier où f ne dépend pas de t, nous avons le théorème :

([1]) H. Bateman, Proc. London Math. Soc., 2e série, t. 7 (1909), t. 8 (1910); H. Bateman, Mathematical analysis of electrical and optical wave-motion (1915), p. 31. Voir aussi G. Darboux, Leçons sur les systèmes orthogonaux (1910), p. 167.

Pour que l'intervalle élémentaire d'un champ statique de gravitation d'un espace vide puisse être représenté par la forme

$$ds^2=f^2(x_1, x_2, x_3)\,(dt^2-dx_1{}^2-dx_2{}^2-dx_3{}^2),$$

il faut et il suffit que f soit une constante.

On trouvera plus loin (§ 14) une application de ce théorème.

Sur les rayons lumineux et les faisceaux naturels des trajectoires.

13. Commençons à trouver les équations différentielles des rayons lumineux au champ *statique* dans l'espace *vide* dont l'intervalle élémentaire est donné par

$$(21)\quad \begin{aligned} ds^2&=f^2dt^2-d\sigma^2\\ &=f^2dt^2-\Sigma g_{ik}\,dx_i\,dx_k \quad (g_{ik}=g_{ki}),\ (i, k=1, 2, 3), \end{aligned}$$

où f et g_{ik} ne dépendent pas de t.

D'après le principe de Fermat(1), les rayons lumineux satisfont à l'équation

$$\delta\int\frac{d\sigma}{f}=0.$$

En appliquant une transformation bien connue(2) et en désignant par $\left\{ {\lambda\mu \atop i} \right\}^*$ le symbole de Christoffel pour la forme $\sum\frac{g_{ik}}{f^2}\,dx_i\,dx_k$, à savoir

$$\begin{aligned} \left\{ {\lambda\mu \atop i} \right\}^* &=\frac{f^2}{2}\sum_{r=1}^{3} g^{ir}\left[\frac{\partial}{\partial x_\mu}\left(\frac{g_{\lambda r}}{f^2}\right)+\frac{\partial}{\partial x_\lambda}\left(\frac{g_{\mu r}}{f^2}\right)-\frac{\partial}{\partial x_r}\left(\frac{g_{\lambda\mu}}{f^2}\right)\right]\\ &=\frac{1}{2}\sum_{r=1}^{3} g^{ir}\left(\frac{\partial g_{\lambda r}}{\partial x_\mu}+\frac{\partial g_{\mu r}}{\partial x_\lambda}-\frac{\partial g_{\lambda\mu}}{\partial x_r}\right)\\ &\qquad -\sum_{r=1}^{3} g^{ir}\left(g_{\lambda r}\frac{\partial \log f}{\partial x_\mu}+g_{\mu r}\frac{\partial \log f}{\partial x_\lambda}-g_{\lambda\mu}\frac{\partial \log f}{\partial x_r}\right)\\ &=\left\{ {\lambda\mu \atop i} \right\}-g_\lambda^i\,\frac{\partial \log f}{\partial x_\mu}-g_\mu^i\,\frac{\partial \log f}{\partial x_\lambda}+g_{\lambda\mu}\sum_{r=1}^{3} g^{ir}\frac{\partial \log f}{\partial x_r}\ (3), \end{aligned}$$

nous trouvons *les équations différentielles des rayons lumineux :*

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(1) Voir Th. de Donder, La gravifique einsteinienne (1921), p. 72.

(2) G. Darboux, Théorie des surfaces, *t.* 2, 2e édition (1915), p. 516; K. Ogura, "Trajectories in the conservative field of force, Part II," Tôhoku Math. Journal, *t.* 8 (1915), p. 188.

(3) Nous employons ici les symboles g^{ir}, $g_\lambda{}^i$, $\left\{ {\lambda\mu \atop i} \right\}$,...., d'après les notations habituelles, pour la forme $d\sigma^2=\Sigma g_{ik}dx_idx_k$.

$$\frac{d^2x_i}{d\sigma^2}+\sum_{\lambda,\,\mu}^{1,2,3}\begin{Bmatrix}\lambda\mu\\ i\end{Bmatrix}^*\frac{dx_\lambda}{d\sigma}\,\frac{dx_\mu}{d\sigma}=0, \qquad (i=1,2,3). \tag{22}$$

Nous pouvons écrire (22) sous la forme :

$$\left\{\begin{aligned}
&dx_1\,d^2x_2-dx_2\,d^2x_1+\begin{Bmatrix}11\\2\end{Bmatrix}^*dx_1^3-\begin{Bmatrix}22\\1\end{Bmatrix}^*dx_2^3\\
&\qquad+\begin{Bmatrix}33\\2\end{Bmatrix}^*dx_1\,dx_3^2-\begin{Bmatrix}33\\1\end{Bmatrix}^*dx_2\,dx_3^2\\
&\quad+\left(2\begin{Bmatrix}12\\2\end{Bmatrix}^*-\begin{Bmatrix}11\\1\end{Bmatrix}^*\right)dx_1^2\,dx_2-\left(2\begin{Bmatrix}12\\1\end{Bmatrix}^*-\begin{Bmatrix}22\\2\end{Bmatrix}^*\right)dx_1dx_2^2\\
&\quad+2\begin{Bmatrix}13\\2\end{Bmatrix}^*dx_1^2\,dx_3-2\begin{Bmatrix}23\\1\end{Bmatrix}^*dx_2^2\,dx_3\\
&\qquad\qquad+2\left(\begin{Bmatrix}23\\2\end{Bmatrix}^*-\begin{Bmatrix}13\\1\end{Bmatrix}^*\right)dx_1\,dx_2\,dx_3=0,\\
&dx_2\,d^2x_3-dx_3\,d^2x_2+\begin{Bmatrix}22\\3\end{Bmatrix}^*dx_2^3-\begin{Bmatrix}33\\2\end{Bmatrix}^*dx_3^3\\
&\qquad+\begin{Bmatrix}11\\3\end{Bmatrix}^*dx_2\,dx_1^2-\begin{Bmatrix}11\\2\end{Bmatrix}^*dx_3\,dx_1^2\\
&\quad+\left(2\begin{Bmatrix}23\\3\end{Bmatrix}^*-\begin{Bmatrix}22\\2\end{Bmatrix}^*\right)dx_2^2\,dx_3-\left(2\begin{Bmatrix}23\\2\end{Bmatrix}^*-\begin{Bmatrix}33\\3\end{Bmatrix}^*\right)dx_2\,dx_3^2\\
&\quad+2\begin{Bmatrix}21\\3\end{Bmatrix}^*dx_1\,dx_2^2-2\begin{Bmatrix}13\\2\end{Bmatrix}^*dx_3^2\,dx_1\\
&\qquad\qquad+2\left(\begin{Bmatrix}13\\3\end{Bmatrix}^*-\begin{Bmatrix}12\\2\end{Bmatrix}^*\right)dx_1\,dx_2\,dx_3=0,\\
&dx_3\,d^2x_1-dx_1\,d^2x_3+\begin{Bmatrix}33\\1\end{Bmatrix}^*dx_3^3-\begin{Bmatrix}11\\3\end{Bmatrix}^*dx_1^3\\
&\qquad+\begin{Bmatrix}22\\1\end{Bmatrix}^*dx_3\,dx_2^2-\begin{Bmatrix}22\\3\end{Bmatrix}^*dx_2^2\,dx_1\\
&\quad+\left(2\begin{Bmatrix}13\\1\end{Bmatrix}^*-\begin{Bmatrix}33\\3\end{Bmatrix}^*\right)dx_3^2\,dx_1-\left(2\begin{Bmatrix}13\\3\end{Bmatrix}^*-\begin{Bmatrix}11\\1\end{Bmatrix}^*\right)dx_3\,dx_1^2\\
&\quad+2\begin{Bmatrix}23\\1\end{Bmatrix}^*dx_2\,dx_3^2-2\begin{Bmatrix}12\\3\end{Bmatrix}^*dx_1^2\,dx_2\\
&\qquad\qquad+2\left(\begin{Bmatrix}12\\1\end{Bmatrix}^*-\begin{Bmatrix}23\\3\end{Bmatrix}^*\right)dx_1\,dx_2\,dx_3=0.
\end{aligned}\right. \tag{23}$$

Pour *le faisceau naturel des trajectoires* ayant la constante d'énergie h [1], nous avons

$$\delta\int\sqrt{\frac{1}{f^2}-h^2}\,d\sigma=0$$ [2].

Par conséquent, si nous prenons la fonction

$$\left(\frac{1}{f^2}-h^2\right)^{-\frac{1}{2}} \text{au lieu de } f$$

dans (22) et (23), nous trouvons des équations correspondantes pour le faisceau naturel des trajectoires.

14. Nous allons maintenant démontrer le théorème suivant :

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(1) P. Painlevé, Journal de Liouville, 4e série, *t.* 10 (1894), p. 5.

(2) H. Weyl, Raum, Zeit, Materie, 4e édition (1921), p. 225.

Si toutes les lignes d'intersection des surfaces appartenant aux trois familles d'un système triplement orthogonal dans l'espace vide sont des rayons lumineux dans un champ statique, cet espace-temps est euclidien.

Prenons les trois familles du système triplement orthogonal comme paramètres x_1, x_2, x_3. L'intervalle élémentaire de ce champ peut se mettre sous la forme

$$ds^2 = f^2\,dt^2 - H_1^2\,dx_1^2 - H_2^2\,dx_2^2 - H_3^2\,dx_3^2,$$

où f, H_1, H_2, H_3 ne dépendent pas de t, et les symboles $\left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\}^*, \ldots$ deviennent

$$\left\{ \begin{matrix} ii \\ i \end{matrix} \right\}^* = \frac{1}{H_i}\frac{\partial H_i}{\partial x_i} - \frac{1}{f}\frac{\partial f}{\partial x_i}, \qquad \left\{ \begin{matrix} ij \\ j \end{matrix} \right\}^* = \frac{1}{H_j}\frac{\partial H_j}{\partial x_i} - \frac{1}{f}\frac{\partial f}{\partial x_i},$$

$$\left\{ \begin{matrix} ii \\ j \end{matrix} \right\}^* = -\frac{H_i}{H_j^2}\frac{\partial H_i}{\partial x_j} + \frac{H_i^2}{H_j^2}\frac{1}{f}\frac{\partial f}{\partial x_j}, \qquad \left\{ \begin{matrix} ij \\ k \end{matrix} \right\}^* = 0, \left(\begin{matrix} i, j, k = 1, 2, 3; \\ i \neq j \neq k \end{matrix} \right).$$

Pour que les intersections de $x_2 = \text{const.}$ et $x_3 = \text{const.}$ soient des rayons lumineux, il faut, en vertue des équations (23), que

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}^* = -\frac{H_1^2}{H_2^2}\left(\frac{1}{H_1}\frac{\partial H_1}{\partial x_2} - \frac{1}{f}\frac{\partial f}{\partial x_2} \right) = 0,$$

$$\left\{ \begin{matrix} 11 \\ 3 \end{matrix} \right\}^* = -\frac{H_1^2}{H_3^2}\left(\frac{1}{H_1}\frac{\partial H_1}{\partial x_3} - \frac{1}{f}\frac{\partial f}{\partial x_3} \right) = 0.$$

De même, pour les intersections $x_3 = \text{const.}$, $x_1 = \text{const.}$ et $x_1 = \text{const.}$, $x_2 = \text{const.}$ nous avons les conditions analogues :

$$\left\{ \begin{matrix} 22 \\ 3 \end{matrix} \right\}^* = -\frac{H_2^2}{H_3^2}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial x_3} - \frac{1}{f}\frac{\partial f}{\partial x_3} \right) = 0,$$

$$\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}^* = -\frac{H_2^2}{H_1^2}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial x_1} - \frac{1}{f}\frac{\partial f}{\partial x_1} \right) = 0;$$

$$\left\{ \begin{matrix} 33 \\ 1 \end{matrix} \right\}^* = -\frac{H_3^2}{H_1^2}\left(\frac{1}{H_3}\frac{\partial H_3}{\partial x_1} - \frac{1}{f}\frac{\partial f}{\partial x_1} \right) = 0,$$

$$\left\{ \begin{matrix} 33 \\ 2 \end{matrix} \right\}^* = -\frac{H_3^2}{H_2^2}\left(\frac{1}{H_3}\frac{\partial H_3}{\partial x_2} - \frac{1}{f}\frac{\partial f}{\partial x_2} \right) = 0.$$

De ces six équations nous trouvons

$$f = \frac{H_1}{\Psi_1(x_1)} = \frac{H_2}{\Psi_2(x_2)} = \frac{H_3}{\Psi_3(x_3)},$$

$\Psi_i(x_i)$ désignant une fonction de la seule variable x_i.

Si nous posons

$$u_1 = \int \Psi_1(x_1)dx_1, \quad u_2 = \int \Psi_2(x_2)dx_2, \quad u_3 = \int \Psi_3(x_3)dx_3,$$

$$f(x_1, x_2, x_3) = \varphi(u_1, u_2, u_3),$$

nous obtenons

$$ds^2 = \varphi^2(u_1, u_2, u_3)\,(dt^2 - du_1^2 - du_2^2 - du_3^2),$$

et le théorème dernier du paragraphe 13 montre que φ doit être une constante.

15. M. W. Blaschke a trouvé l'expression de *la courbure* $\frac{1}{\rho}$ *de la courbe*

$$x_1 = x_1(\sigma), \qquad x_2 = x_2(\sigma), \qquad x_3 = x_3(\sigma)$$

dans l'espace riemannien (¹). D'après lui, la courbure $\frac{1}{\rho}$ pour l'espace ayant l'élément linéaire

$$d\sigma^2 = H_1^2\,dx_1^2 + H_2^2\,dx_2^2 + H_3^2\,dx_3^2$$

est donnée par la formule

$$\frac{1}{\rho^2} = (H_1^2 B_1^2 + H_2^2 B_2^2 + H_3^2 B_3^2) - \left(H_1^2 B_1 \frac{dx_1}{d\sigma} + H_2^2 B_2 \frac{dx_2}{d\sigma} + H_3^2 B_3 \frac{dx_3}{d\sigma}\right)^2, \tag{24}$$

où

$$\begin{aligned}
B_1 = \frac{d^2x_1}{d\sigma^2} &+ \frac{1}{H_1}\frac{\partial H_1}{\partial x_1}\left(\frac{dx_1}{d\sigma}\right)^2 + \frac{2}{H_1}\frac{\partial H_1}{\partial x_1}\frac{dx_1}{d\sigma}\frac{dx_2}{d\sigma} + \frac{2}{H_1}\frac{\partial H_1}{\partial x_3}\frac{dx_1}{d\sigma}\frac{dx_3}{d\sigma} \\
&- \frac{H_2}{H_1^2}\frac{\partial H_2}{\partial x_1}\left(\frac{dx_2}{d\sigma}\right)^2 - \frac{H_3}{H_1^2}\frac{\partial H_3}{\partial x_1}\left(\frac{dx_3}{d\sigma}\right)^2, \\
B_2 = \frac{d^2x_2}{d\sigma^2} &+ \frac{1}{H_2}\frac{\partial H_2}{\partial x_2}\left(\frac{dx_2}{d\sigma}\right)^2 + \frac{2}{H_2}\frac{\partial H_2}{\partial x_3}\frac{dx_2}{d\sigma}\frac{dx_3}{d\sigma} + \frac{2}{H_2}\frac{\partial H_2}{\partial x_1}\frac{dx_3}{d\sigma}\frac{dx_1}{d\sigma} \\
&- \frac{H_3}{H_2^2}\frac{\partial H_3}{\partial x_2}\left(\frac{dx_3}{d\sigma}\right)^2 - \frac{H_1}{H_2^2}\frac{\partial H_1}{\partial x_2}\left(\frac{dx_1}{d\sigma}\right)^2, \\
B_3 = \frac{d^2x_3}{d\sigma^2} &+ \frac{1}{H_3}\frac{\partial H_3}{\partial x_3}\left(\frac{dx_3}{d\sigma}\right)^2 + \frac{2}{H_3}\frac{\partial H_3}{\partial x_1}\frac{dx_3}{d\sigma}\frac{dx_1}{d\sigma} + \frac{2}{H_3}\frac{\partial H_3}{\partial x_2}\frac{dx_3}{d\sigma}\frac{dx_2}{d\sigma} \\
&- \frac{H_1}{H_3^2}\frac{\partial H_1}{\partial x_3}\left(\frac{dx_1}{d\sigma}\right)^2 - \frac{H_2}{H_3^2}\frac{\partial H_2}{\partial x_3}\left(\frac{dx_2}{d\sigma}\right)^2.
\end{aligned}$$

Pour les rayons lumineux, en tenant compte des équations (22) (²),

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(¹) W. Blaschke, "Frenets Formeln für den Raum von Riemann," Math. Zeitschrift, t. 6 (1920), p. 94. Je ne considère pas ici la *torsion* de la courbe.

(²) Dans ce cas les équations (22) deviennent

$$\begin{aligned}
\frac{d^2x_1}{d\sigma^2} &+ \left(\frac{1}{H_1}\frac{\partial H_1}{\partial x_1} - \frac{1}{f}\frac{\partial f}{\partial x_1}\right)\left(\frac{dx_1}{d\sigma}\right)^2 + 2\left(\frac{1}{H_1}\frac{\partial H_1}{\partial x_2} - \frac{1}{f}\frac{\partial f}{\partial x_2}\right)\frac{dx_1}{d\sigma}\frac{dx_2}{d\sigma} \\
&+ 2\left(\frac{1}{H_1}\frac{\partial H_1}{\partial x_3} - \frac{1}{f}\frac{\partial f}{\partial x_3}\right)\frac{dx_1}{d\sigma}\frac{dx_3}{d\sigma} \\
&+ \left(-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial x_1} + \frac{H_2^2}{H_1^2}\frac{1}{f}\frac{\partial f}{\partial x_1}\right)\left(\frac{dx_2}{d\sigma}\right)^2 + \left(-\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial x_1} + \frac{H_3^2}{H_1^2}\frac{1}{f}\frac{\partial f}{\partial x_1}\right)\left(\frac{dx_3}{d\sigma}\right)^3 = 0.
\end{aligned}$$

nous avons

$$B_1 = \frac{1}{f}\frac{\partial f}{\partial x_1}\left(\frac{dx_1}{d\sigma}\right)^2 + \frac{2}{f}\frac{\partial f}{dx_2}\frac{dx_1}{d\sigma}\frac{dx_2}{d\sigma} + \frac{2}{f}\frac{\partial f}{dx_3}\frac{dx_1}{d\sigma}\frac{dx_3}{d\sigma}$$

$$- \frac{1}{H_1^2}\frac{1}{f}\frac{\partial f}{\partial x_1}\left[H_2^2\left(\frac{dx_2}{d\sigma}\right)^2 + H_3^2\left(\frac{dx_3}{d\sigma}\right)^2\right]$$

$$= \frac{2}{f}\frac{dx_1}{d\sigma}\left(\frac{\partial f}{\partial x_1}\frac{dx_1}{d\sigma} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{d\sigma} + \frac{\partial f}{\partial x_3}\frac{dx_3}{d\sigma}\right) - \frac{1}{f}\frac{\partial f}{\partial x_1}\left(\frac{dx_1}{d\sigma}\right)^2$$

$$- \frac{1}{H_1^2}\frac{1}{f}\frac{\partial f}{\partial x_1}\left[1 - H_1^2\left(\frac{dx_1}{d\sigma}\right)^2\right]$$

$$= \frac{2}{f}\frac{df}{d\sigma}\frac{dx_1}{d\sigma} - \frac{1}{H_1^2}\frac{1}{f}\frac{\partial f}{\partial x_1},$$

$$B_2 = \frac{2}{f}\frac{df}{d\sigma}\frac{dx_2}{d\sigma} - \frac{1}{H_2^2}\frac{1}{f}\frac{\partial f}{\partial x_2},$$

$$B_3 = \frac{2}{f}\frac{df}{d\sigma}\frac{dx_3}{d\sigma} - \frac{1}{H_3^2}\frac{1}{f}\frac{\partial f}{\partial x_3}.$$

Par conséquent, nous obtenons *l'expression suivante pour la courbure du rayon lumineux :*

$$(25)_1 \quad \frac{1}{\rho^2} = \frac{1}{H_1^2}\left(\frac{1}{f}\frac{\partial f}{\partial x_1}\right)^2 + \frac{1}{H_2^2}\left(\frac{1}{f}\frac{\partial f}{\partial x_2}\right)^2 + \frac{1}{H_3^2}\left(\frac{1}{f}\frac{\partial f}{\partial x_3}\right) - \left(\frac{1}{f}\frac{df}{d\sigma}\right)^2.$$

Si nous employons le paramètre différentiel du premier ordre de Lamé, $(25)_1$ devient

$$(25)_2 \qquad \frac{1}{\rho^2} = \Delta(\log f) - \left(\frac{d\log f}{d\sigma}\right)^2.$$

On peut la mettre encore sous la forme :

$$(25)_3 \qquad \frac{1}{\rho^2} = \left(\frac{\partial \log f}{\partial n}\right)^2 - \left(\frac{d\log f}{d\sigma}\right)^2,$$

$\dfrac{\partial \log f}{\partial n}$ désignant la dérivée de $\log f$ prise suivant la normale à la surface $\log f = \text{const.}$

Nous voyons que les formules $(25)_2$ et $(25)_3$ ne dépendent pas du choix des coordonnées x_1, x_2, x_3 : en d'autres termes, *les formules* $(25)_2$ *et* $(25)_3$ *sont valables pour l'espace-temps* (21).

16. Cherchons maintenant la variation de cette courbure pour les directions variables autour d'un point fixe. Nous trouvons facilement :

$$\left(\frac{1}{\rho}\right)^2_{min} = 0 \qquad \text{(suivant la normale à la surface } f = \text{const.)},$$

$$\left(\frac{1}{\rho}\right)^2_{max}=\left(\frac{\partial \log f}{\partial n}\right)^2 \text{ (suivant toutes les tangentes à la surface } f=\text{const.}).$$

Au point de vue purement mathématique, il est intéressant d'étudier *l'ensemble des courbes dont les directions sont telles que la difference*

$$\left(\frac{1}{\rho}\right)^2_{max}-\left(\frac{1}{\rho}\right)^2$$

est constante.

Les courbes ainsi définies satisfont à l'équation de Monge:

$$(26)\quad \left(\frac{\partial \log f}{\partial x_1}dx_1+\frac{\partial \log f}{\partial x_2}dx_2+\frac{\partial \log f}{\partial x_3}dx_3\right)^2 =k^2(H_1^2\,dx_1^2+H_2^2\,dx_2^2+H_3^2\,dx_3^2),$$

k étant une constante indépendante de x_1, x_2, x_3.

Puisque nous pouvons écrire cette équation sous la forme

$$(27)\qquad k\int_{x_1{}^0 x_2{}^0 x_3{}^0}^{x_1{}^1 x_2{}^1 x_3{}^1} d\sigma=(\log f)_{x_1{}^1 x_2{}^1 x_3{}^1}-(\log f)_{x_1{}^0 x_2{}^0 x_3{}^0},$$

l'ensemble des courbes forme un système de courbes sans détours au sens généralisé.

Sur les surfaces paramétriques, par exemple sur $x_3=$const., nous avons *le système de courbes sans détours* "Kurvennetz ohne Umwege" au sens de M. G. Scheffers (1):

$$\left(\frac{\partial \log f}{\partial x_1}dx_1+\frac{\partial \log f}{\partial x_2}dx_2\right)^2=k^2(H_1^2\,dx_1^2+H_2^2\,dx_2^2).$$

17. Si nous prenons $\left(\frac{1}{f^2}-h^2\right)^{-\frac{1}{2}}$ au lieu de f dans (25), (26) et (27), nous trouvons des équations correspondantes pour le faisceau naturel des trajectoires ayant la constante d'énergie h.

Il est facile de démontrer qu'*à chaque point de l'espace la valeur absolue de la courbure d'un rayon lumineux n'est pas plus grande que celle de la trajectoire à la même direction.* En effet, de $(25)_3$, nous avons

$$\left(\frac{1}{\rho}\right)^2_{traj.}=\frac{1}{(1-h^2f^2)^2}\left[\left(\frac{\partial \log f}{\partial n}\right)^2-\left(\frac{d \log f}{d\sigma}\right)^2\right]$$

(1) Pour le système de courbes sans détours sur un plan ou sur une surface dans l'espace euclidien, voir G. Scheffers, Leipziger Berichte, t. 57 (1905), p. 353; R. v. Lilienthal, Vorlesungen über Differentialgeometrie, t. 1 (1908), p. 123; t. 2 (1913), p. 242; K. Ogura, Proceedings of Tôkyô Math.-Physical Society, 2e série, t. 9 (1918), p. 284, p. 409. Voir aussi M. d'Ocagne, Bull. Soc. Math. France, t. 13 (1885), p. 71.

$$=\frac{1}{(1-h^2f^2)^2}\cdot\left(\frac{1}{\rho}\right)^2_{lumière},$$

d'où

$$\left(\frac{1}{\rho}\right)_{traj.}=\pm\frac{1}{1-h^2f^2}\cdot\left(\frac{1}{\rho}\right)_{lumière.}$$

Puisque nous pouvons considérer le rayon lumineux comme limite de la trajectoire dont la constante h tend vers nulle, nous trouvons *la relation remarquable :*

$$(28)\quad \left(\frac{1}{\rho}\right)_{traj.}=\frac{1}{1-h^2f^2}\cdot\left(\frac{1}{\rho}\right)_{lumière,}\qquad (1-h^2f^2>0).$$

Paris, le 28 octobre 1921.

J'ai appris que la théorème du paragraphe 12 a été déjà trouvé par M. E. Kasner dans son Mémoire "Einstein theory of gravitation : Determination of the field by light signals," American Journal of Mathematics, Vol. 43, No. 1 (1921), p. 20. Au sujet de la même nature, voir aussi Kasner, Math. Ann., 85 (1922), p. 227 ; J. A. Schoute et D. J. Struik, Amer. Jour. of Math., Vol. 43, No. 4 (1922), p. 213 : Kasner, ibid, p. 217.

Ôsaka, le 15 juillet 1922.

K. O.

Untersuchungen über das Poissonsche Integral auf der Kugel und seine Ableitungen,

von

P. Edwin Strässle, in Stans (Schweiz).

Einleitung.

Wir bezeichnen mit S die Oberfläche einer Kugel, deren Zentrum C ist. Der Einfachheit wegen wählen wir als Längeneinheit den Radius dieser Kugel. K sei das Innere dieser Kugel mit Ausschluss der Punkte von S. Die kleinen Buchstaben $p, q, p' \ldots$ bedeuten Punkte von K, die von C verschieden sind, die grossen Buchstaben $P, Q, P' \ldots$ Punkte von S. Derselbe grosse und kleine Buchstabe, z. B. p, P, bezeichnet zwei Punkte, welche auf demselben Radius CP liegen. Der Abstand des Punktes p von den Punkten q, P', wird dargestellt durch $\overline{pq}$, $\overline{pP'} \ldots$ und der Winkel PCP' durch $\omega(P, P')$. Weil der Kugelradius gleich der Längeneinheit ist, so misst ω zugleich den sphärischen Abstand zwischen P und P'.

Ist $V(P)$ eine für jeden Punkt P von S definierte und auf S im Sinne von Lebesgue integrierbare Funktion, so bezeichnen wir mit

$$\int_S V(P')d\sigma_{P'} \qquad \int_{\mathfrak{M}} V(P)d\sigma_P$$

die Flächenintegrale von $V(P)$ über S, bezw. über eine messbare Punktmenge $\mathfrak{M}$ von S; $d\sigma_P$ bedeutet dabei das Flächenelement im Punkte P. Sehr oft werden wir Gebiete zu betrachten haben, welche aus allen Punkten einer Kalotte A mit Zentrum P und sphärischem Radius $\varepsilon (0 < \varepsilon < \pi)$ oder der zugehörigen Komplementärkalotte $S-A$ gebildet werden. Wir stellen dann durch

$$\int_{\omega(P,\,P') \leq \varepsilon} V(P')d\sigma_{P'}, \qquad \int_{\omega(P,\,P') > \varepsilon} V(P')d\sigma_{P'}.$$

die Integrale über diese beiden Kalotten dar. Offenbar gilt

$$\int_S V(P')d\sigma_{P'} = \int_{\omega(P,\,P') \leq \varepsilon} V(P')d\sigma_{P'} + \int_{\omega(P,\,P') > \varepsilon} V(P')d\sigma_{P'}.$$

Wir setzen einzig voraus, dass $V(P')$ auf S im Sinne von Lebesgue integrierbar sei. Man kann sich leicht überzeugen, dass dann die Funktion

$$v(p)=\frac{1}{4\pi}\int\limits_S\frac{1-\overline{pC}^2}{\overline{pP'}^3}\ V(P')d\sigma_{P'}=\frac{1-\overline{pC}^2}{4\pi}\int\limits_S\frac{V(P')}{\overline{pP'}^3}d\sigma_{P'}\quad(1)$$

existiert und nebst allen ihren Ableitungen in jedem Punkte p von K stetig ist. Die Beziehung

$$\Delta\left(\frac{1-\overline{pC}^2}{\overline{pP'}^3}\right)=0,$$

wo Δ die Operation von Laplace bedeutet, zeigt, dass $v(p)$ in K eine Lösung der Laplaceschen Gleichung $\Delta\ v(p)=0$ darstellt.

Das Integral (1) trägt den Namen *Integral von Poisson*. Das Verhalten von $v(p)$, wenn p sich einem Punkte P_0 von S nähert, ist bereits Gegenstand zahlreicher Untersuchungen geworden (¹). In dem besonderen Falle, wo $V(P)$ eine auf S stetige Funktion ist, strebt, wie H. A. Schwarz (²) gezeigt hat, $v(p)$ gegen $V(P_0)$, wenn p sich P_0 nähert. Diese wichtige Eigenschaft zeigt, dass die Funktion $v(p)$ in diesem Falle eine Lösung des Dirichlet'schen Problems für die Kugel darstellt. Es ist deshalb von Interesse zu sehen, inwieweit $v(p)\rightarrow V(P_0)$ für $p\rightarrow P_0$ noch gilt, wenn man für $V(P)$ nur die Integrierbarkeit im Sinne von Lebesgue voraussetzt. Diese Untersuchung bildet den Gegenstand des *ersten Kapitels* dieser Arbeit. Die Resultate, zu denen wir gelangen, sind denjenigen analog, welche P. Fatou in einer hervorragenden Arbeit der Acta Mathematica (³) (in bezug auf die Ebene) veröffentlicht hat. Ferner decken sich die Resultate zum Teil mit den Sätzen, welche G. Julia (⁴) neulich auf anderem Wege gefunden hat. Dazu entwickeln wir einige Sätze neuer Art. (Theorem V, VI u. VII).

Im *zweiten Kapitel* wird die *Einzigkeit* der Lösung des Dirichlet'schen Problems in bezug auf die Kugel behandelt und ein Kriterium der Unität, welches M. Plancherel (⁵) für dasselbe Problem in der Ebene auf-

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(1) Vgl. Enzyclopädie der math. Wissensch. II A 7 b (H. Burkhardt und Fr. Meyer), p. 480; IIC3 (L. Lichtenstein), p. 220.

(2) H. A. Schwarz, Ges. Abh. 2 p. 144-171 und 175-210. Vgl. E. Picard, Traité d'analyse, 2. Aufl. 1, p. 164.

(3) P. Fatou, Séries trigonométriques et séries de Taylor [Acta Mathematica 30 (1906), p. 335-400].

(4) Gaston Julia, Sur les valeurs limites de l'integrale de Poisson relative à la sphère en un point de discontinuité des données. [Bull. des Sc. math. (2) 42 (1918), p. 214-220 und 224-231].

(5) M. Plancherel, Remarques sur l'intégration de l'equation $\Delta u=0$ [Bull. des Sc. math. (2) 34(1910), p. 111-114] Vgl. O. D. Kellogg, Harmonic functions und Green's integral [Transact. of the Americ. soc. 12 (1912), p. 109-122 und Encyclopädie II C 3, p. 219]

gestellt hat, auf die Kugel übertragen. Da die Note, in welcher Herr Plancherel seine Resultate veröffentlicht hat, sehr kurz gehalten und folglich nicht leicht zu lesen ist, haben wir es für gut gehalten, die Entwicklung ausführlich darzustellen.

Das *dritte und vierte Kapitel* beschäftigen sich mit dem Verhalten der *Ableitungen von* $v(p)$, wenn p sich P_0 nähert. Führen wir, um uns genauer auszudrücken, Polarkoordinaten mit Ursprung in C ein. Seien r, ϑ, φ die Polarkoordinaten von p, ferner 1, ϑ', φ' diejenigen von P' und 1, ϑ_0, φ_0 diejenigen von P_0. $v(p)$ wird dann eine Funktion $v(r, \vartheta, \varphi)$ und $V(P')$ eine Funktion $V(\vartheta',\varphi')$. Im dritten und vierten Kapitel werden nun folgende Sätze entwickelt werden:

Wenn $V(P)$ *im Punkte* P *ein Differential der Ordnung* $h+k$ *besitzt, so gilt*

$$\lim_{(r,\vartheta,\varphi)\to(1,\vartheta_0,\varphi_0)} \frac{\delta^{h+k} v(r,\vartheta,\varphi)}{\delta\vartheta^h \delta\varphi^k} = \lim_{p\to P_0} \frac{\delta^{h+k} v(p)}{\delta\vartheta^h\delta\varphi^k} = \frac{\delta^{h+k} V(P_0)}{\delta\vartheta^h\delta\varphi^k},$$

vorausgesetzt, dass der Weg in K, *auf welchem* p *sich* P_0 *nähert, in einem räumlichen Winkel eingeschlossen bleibt, dessen Scheitel in* P_0 *liegt und welcher* S *nicht berührt.*

Wenn $V(P)$ *im Punkte* P_0 *und in der Umgebung des Punktes* P_0 *ein Differential der Ordnung* $h+k$ *besitzt und dieses Differential im Punkte* P_0 *stetig ist, so gilt*

$$\lim_{p\to P_0} \frac{\delta^{h+k} v(p)}{\delta\vartheta^h\delta\varphi^k} = \frac{\delta^{h+k} V(P_0)}{\delta\vartheta^h\delta\varphi^k},$$

wenn p *sich auf einem beliebigen in* K *gelegenen Wege* P_0 *nähert.*

Die gleiche Frage ist bereits von P. Fatou (¹) und vollständiger von Ch.-J. de la Vallée Poussin (²) für das Integral von Poisson längs eines Kreises behandelt worden. Die von diesen Autoren angewandten Methoden lassen sich nicht auf den Raum übertragen. Man könnte das Problem unmittelbar durch Berechnung der einzelnen Ableitungen lösen. Dieser Weg führt zu sehr allgemeinen Resultaten, in welchen die verallgemeinerten Ableitungen vorkommen(³), verliert sich aber rasch

(¹) Siehe die in Fussnote 3 zitierte Arbeit. Fatou behandelt nur die erste und zweite Ableitung.

(²) Ch. J. de la Vallée Poussin, Sur l'approximation des fonctions d'une variable réelle et de leurs dérivees par des polynômes et des suites limitées de Fourier. Bull. de la classe des Sciences de l'Académie Royale de Belgique (1908). p. 245-254. Herr de la Vallée Poussin behandelt nur die radiale Annäherung.

(³) Diese Resultate sind in § 4 des vierten Kapitels dargestellt.

in äusserst komplizierten Rechnungen. Wir mussten deshalb eine andere Methode ausfindig machen. Sie besteht darin, zuerst das Verhalten der Ableitungen in dem besonderen Falle zu studieren, wo $V(P)$ analytisch ist, dann im allgemeinen Falle $V(P)$ mit Hilfe der Taylorschen Formel in einen analytischen Teil und einen Rest zu zerlegen und endlich zu zeigen, dass die auf den Rest bezüglichen Teile des Poissonschen Integrals verchwindend klein werden.

Die Bedeutung der vorliegenden Untersuchungen erhellt besonders aus den Beziehungen des Poissonschen Integrals zur *Reihe von Laplace*. Wie das Poissonsche Integral in bezug auf den Kreis ein Mittel bildet, die Summation einer nicht konvergenten oder konvergenten Reihe von Fourier durchzuführen, so leistet das Poissonsche Integral in bezug auf die Kugel denselben Dienst für die Reihe von Laplace ([1]).

Vorliegende Arbeit ist ein Auszug aus meiner der naturwiss. Fakultät der Universität Freiburg (Schweiz) im März 1921 vorgelegten Inaugural-Dissertation.

Erstes Kapitel.

Verhalten des Poissonschen Integrals $v(P)$ bei Annäherung von p an einen Punkt P_0 der Kugeloberfläche.

§ 1. Obere und untere Grenze des Poissonschen Integrals für $p \to P_0$.

Wenn (r, ϑ, φ) und $(1, \vartheta', \varphi')$ die Polarkoordinaten von p und P' sind, so ist der Abstand $\overline{pP'}$ in Formel (1) nach dem Kosinussatz ausgedrückt durch die Gleichung

$$\overline{pP'}^2 = 1 - 2\overline{pC}\cos\omega + \overline{pC}^2 = 1 - 2r\cos\omega + r^2,$$

wo

$$\cos\omega = \cos\vartheta\,\cos\vartheta' + \sin\vartheta\,\sin\vartheta'\,\cos(\varphi - \varphi').$$

Der Diskontinuitätsfaktor $\dfrac{1-\overline{pC}^2}{\overline{pP'}^3}$ ist also eine Funktion der Strecke $\overline{pC} = r$ und des Winkels ω (P, P'). Nennen wir deshalb

$$\frac{1-\overline{pC}^2}{\overline{pP'}^3} = \frac{1-r^2}{(1-2r\cos\omega + r^2)^{3/2}} = F(r, \omega).$$

Das Poissonsche Integral bekommt nun die Form

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(1) Vgl. E. Picard, Traité d'analyse, 1, p. 279.–Ch. J. de la Vallée Poussin, loco citato, p. 245.–Encyklopädie IIC3 (L. Lichtenstein), p. 225.

$$v(p) = \frac{1}{4\pi}\int_S F(r, \omega)\, V(P')\, d\sigma_{P'}.$$

Dieses Integral ist von der Art und Lage des Koordinatensystems unabhängig. Bestimmen wir seinen Wert, wenn $V(P)$ den konstanten Wert 1 hat. Wir wählen das Polarkoordinatensystem so, dass p sich auf dem positiven Teil der Polachse befindet. Dann ist $\vartheta = 0$; $\omega = \vartheta'$; $d\sigma_{P'} = \sin\vartheta' d\vartheta' d\varphi'$. Ferner folgt dann aus

$$\overline{pP'}^2 = 1 - 2r\cos\vartheta' + r^2 \equiv r_1^2,$$

dass

$$\sin\vartheta' d'\vartheta' = \frac{r_1 dr_1}{r},$$

und wir erhalten für jeden Punkt p von K

$$\frac{1}{4\pi}\int_S F(r,\omega)\, d\sigma_{P'} = \frac{1}{4\pi}\int_0^\pi F(r,\vartheta')\sin\vartheta' d\vartheta' \int_0^{2\pi} d\varphi' = \frac{1-r^2}{2r}\int_{1-r}^{1+r}\frac{dr_1}{r_1^2} = 1. \tag{2}$$

Die nun folgenden Sätze haben die einzige allgemeine *Voraussetzung*, dass $V(P)$ auf S im Sinne von Lebesgue integrierbar ist, d.h. dass es eine positive endliche Zahl M gibt, derart dass

$$\int_S |V(P)|\, d\sigma_P \leqq M.$$

Theorem I. Das Verhalten des Poissonschen Integrals $v(p)$ bei Annäherung $p \to P_0$ auf beliebigem Wege in K ist nur abhängig von den Werten von $V(P)$ in der Umgebung ([1]) *des Punktes P_0.* In einer Formel:

$$\lim_{p \to P_0} \frac{1}{4\pi}\int_{\omega(P_0, P') > \varepsilon} F(r,\omega)\, V(P')\, d\sigma_P = 0 \tag{3a}$$

oder

$$\lim_{p \to P_0} \left[v(p) - \frac{1}{4\pi}\int_{\omega(P_0, P') \leqq \varepsilon} F(r, \omega)\, V(P')\, d\sigma_{P'} \right] = 0. \tag{3b}$$

In Wirklichkeit lässt sich noch mehr zeigen, nämlich, dass die Grenzwerte (3*a*) und (3*b*) gleichmässig in P_0 *auf der ganzen Kugel* gelten, denn

$$\left| \frac{1}{4\pi}\int_{\omega(P_0, P') > \varepsilon} F(r, \omega)\, V(P')\, d\sigma_P \right| \leqq \frac{1}{4\pi} \max_{\varepsilon \leqq \omega \leqq \pi} | F(r,\omega) | \int_S | V(P') |\, d\sigma_{P'}$$

([1]) Unter Umgebung ε des Punktes P_0 verstehen wir die Menge aller Punkte P', welche der Beziehung $o < \omega(P_0, P') \leqq \varepsilon$ genügen, wo ε eine beliebige positive Zahl $< \pi$ ist.

$$\leqq \frac{M}{4\pi}(1-r^2) \max_{\varepsilon \leqq \omega \leqq \pi} \frac{1}{\overline{pP'}^3}$$

$$\leqq \frac{M}{4\pi}(1-r^2) \max_{\varepsilon \leqq \omega \leqq \pi} \frac{1}{|\overline{P_0P'}-\overline{P_0p}|^3}.$$

Wenn also $\overline{P_0p} < \sin\frac{\varepsilon}{2}$, so ist $|\overline{P_0P'}-\overline{P_0p}| > 2\sin\frac{\varepsilon}{2} - \sin\frac{\varepsilon}{2} = \sin\frac{\varepsilon}{2}$

und folglich $\left|\frac{1}{4\pi}\int_{\omega(P_0,\,P')>\varepsilon} F(r,\omega)\,V(P')\,d\sigma_{P'}\right| \leqq \frac{M}{4\pi \sin^3\frac{\varepsilon}{2}}(1-r^2).$ (4a)

Aus diesen Abschätzungen, ergibt sich ferner, dass für irgendein messbares Gebiet $\mathfrak{M}$ das der Kalotte $\omega(P_0, P') > \varepsilon$ angehört, noch gilt

$$\frac{1}{4\pi}\int_{\mathfrak{M}} F(r,\omega)\ |\ V(P')\ |\ d\sigma_{P'} \leqq G(\varepsilon)(1-r^2), \tag{4b}$$

wo $G(\varepsilon)$ eine nur von $|\varepsilon|$ abhängige endliche Zahl ist.

Aus den Formeln (2) und (3) folgt unmittelbar

$$\lim_{p\to P_0} \frac{1}{4\pi}\int_{\omega(P_0,\,P')\leqq\varepsilon} F(r,\,\omega)\,d\sigma_{P'} = \lim_{p\to P_0} \frac{1}{4\pi}\left[\int_S F(r,\omega)\,d\sigma_{P'} - \int_{\omega(P_0,\,P')>\varepsilon} F(r,\omega)\,d\sigma_{P'}\right] = 1. \tag{5}$$

Nehmen wir nun an, die Randfunktion $V(P)$ besitze in der Umgebung ε des Punktes P_0 eine obere Schranke $M(\varepsilon)$ und eine untere Schranke $m(\varepsilon)$. Weil $F(r,\omega)$ *immer positiv bleibt*, so lässt sich der Mittelwertsatz auf $v(p)$ anwenden und wir erhalten

$$m(\varepsilon)\frac{1}{4\pi}\int_{\omega(P_0,\,P')\leqq\varepsilon} F(r,\,\omega)\,d\sigma_{P'} \leqq \frac{1}{4\pi}\int_{\omega(P_0,\,P')\leqq\varepsilon} F(r,\,\omega)\,V(P')\,d\sigma_{P'} \leqq M(\varepsilon)\frac{1}{4\pi}\int_{\omega(P_0,\,P')\leqq\varepsilon} F(r,\,\omega)\,d\sigma_{P'}$$

Unter Berücksichtigung von (3) und (4) folgt daraus

$$m(\varepsilon) \leqq \liminf_{p\to P_0} v(p) \leqq \limsup_{p\to P_0} v(p) \leqq M(\varepsilon). \tag{6}$$

Lässt man jetzt ε gegen Null streben, so bekommt man

Theorem II. Die obere und untere Grenze des Poissonschen Integrals $v(p)$ bei Annäherung $p\to P_0$ auf beliebigem Wege in K sind enthalten zwischen der obern und untern Limesfunktion von $V(P)$ im Punkte P_0.

Zusatz. Ist $V(P)$ auf der ganzen Kugelfläche beschränkt, so liegen alle Werte von $v(p)$ zwischen der obern und untern Schranke von $V(P)$, denn, ist $M(S)$ die obere und $m(S)$ die untere Schranke von $V(P)$ auf S, so ergibt die Anwendung des Mittelwertsatzes

$$\frac{m(S)}{4\pi}\int_S F(r,\,\omega)\,d\sigma_{P'} \leqq \frac{1}{4\pi}\int_S F(r,\,\omega)\,V(P')\,d\sigma_{P'} \leqq \frac{M(S)}{4\pi}\int_S F(r,\omega)\,d\sigma_{P'},$$

woraus unter Berücksichtigung von Formel (2) der angeführte Zusatz sich ergibt.

§ 2. Die Randfunktion ist stetig in P_0.

Setzen wir die Stetigkeit von $V(P)$ im Punkte P_0 voraus, so dass also

$$\lim_{\varepsilon \to 0} m(\varepsilon) = \lim_{\varepsilon \to 0} M(\varepsilon) = V(P_0),$$

so folgt aus (6)

$$\lim_{p \to P_0} v(p) = V(P_0).$$

Ist $V(P)$ in allen Punkten im Innern einer Kalotte stetig, so gilt, wie aus dem vorigen hervorgeht, der letzte Grenzwert gleichmässig innerhalb dieser Kalotte ([1]). Wir haben also

Theorem III. Das Poissonsche Integral $v(p)$ strebt bei beliebiger Annäherung $p \to P_0$ in K gegen einen Stetigkeitspunkt P_0 der Randfunktion $V(P)$ dem Werte $V(P_0)$ zu. Es strebt gleichmässig gegen diesen Wert innerhalb einer Kalotte, auf welcher $V(P)$ überall stetig ist.

§ 3. Die Randfunktion ist unstetig im Punkte P_0, besitzt aber in diesem Punkte einen Kreis- bezw. Kalotten-mittelwert.

Sei P_0 der Pol eines Kugelkreises mit dem sphärischen Abstand t; der Radius des Kreises ist $\sin t$; das Bogenelement im Punkte P' des Kreises sei $ds_{P'}$. Wenn das längs dieses Kreises genommene Linienintegral

$$V_1(P_0\,;\,t) = \frac{1}{2\pi \sin t} \int_{\omega(P_0,\,P')=t} V(P')\,ds_{P'}$$

existiert ([2]) und für $t \to 0$ einen Grenzwert

$$V_1(P_0) = \lim_{t \to 0} V_1(P_0\,;\,t)$$

hat, so nennen wir $V_1(P_0)$ den *Kreismittelwert* von $V(P)$ in P_0.

Theorem IV. Existiert in einem Punkte P von S der Kreismittelwert von $V(P)$

$$V_1(P_0) = \lim_{t \to 0} \frac{1}{2\pi \sin t} \int_{\omega(P_0,P')=t} V(P')\,ds_{P'},$$

so strebt das Poissonsche Integral $v(p)$ bei radialer Annäherung $p \to P_0$ gegen diesen Mittelwert.

([1]) Unter Gleichmässigkeit im Inneren eines Bereiches verstehen wir die gewöhnliche Gleichmässigkeit in jedem abgeschlossenen Teilbereich.

([2]) Aus der Integrierbarkeit von $V(P)$ im Sinne von Lebesgue folgt nur, dass die Werte t, für welche $V_1(P_0\,;\,t)$ nicht existiert, eine Menge vom Masse Null bilden.

Zur Vereinfachung des Beweises wählen wir das Achsensystem so, dass P_0 in den Pol liegen kommt und P_0 sich längs der Polachse bewegt. Es ist dann $\vartheta'=t$, $V(P')=V(t, \varphi')$ und $F(r, \omega)=F(r, t)$ ist von φ' unabhängig. Der Kreismittelwert hat dann die Form

$$V_1(P_0)=\lim_{t\to o} V_1(P_0;\ t)=\lim_{t\to o}\frac{1}{2\pi}\int_0^{2\pi} V(t,\varphi')d\varphi'. \tag{7}$$

Für das Poissonsche Integral erhalten wir

$$v(p_0)=\frac{1}{2}\int_0^{\pi} F(r,t)\sin t\,dt\ \frac{1}{2\pi}\int_0^{2\pi} V(t,\varphi')d\varphi'$$

$$=\frac{1}{2}\int_0^{\pi} V_1(P_0;\ t)F(r,t)\sin t\,dt.$$

Nach Formel (3*b*) wird also

$$\lim_{p_0\to P_0}\left[v(p_0)-\frac{1}{2}\int_0^{\varepsilon} V_1(P_0;\ t)F(r,t)\sin t\,dt\right]=0. \tag{8}$$

Wenn nun auf der Kalotte $\omega(P_0, P')\leqq\varepsilon$ eine obere Schranke $\overline{M}(\varepsilon)$ und eine untere Schranke $\overline{m}(\varepsilon)$ von $V_1(P_0;\ t)$ existiert, so ergibt sich nach dem Mittelwertsatz für das letzte Integral

$$\frac{\overline{m}(\varepsilon)}{2}\int_0^{\varepsilon} F(r,t)\sin t\,dt\leqq\frac{1}{2}\int_0^{\varepsilon} V_1(P_0;\ t)F(r,t)\sin t\,dt$$

$$\leqq\frac{\overline{M}(\varepsilon)}{2}\int_0^{\varepsilon} F(r,t)\sin t\,dt.$$

Aus (5) folgt aber

$$\lim_{p_0\to P_0}\frac{1}{4\pi}\int_{\omega(P_0, P')\leqq\varepsilon} F(r,t)\,d\sigma_{P'}=\lim_{p_0\to P_0}\frac{1}{2}\int_0^{\varepsilon} F(r,t)\sin t\,dt=1. \tag{9}$$

Nach den letzten drei Gleichungen ist

$$\overline{m}(\varepsilon)\leqq\liminf_{p_0\to P_0} v(p_0)\leqq\limsup_{p_0\to P_0} v(p_0)\leqq\overline{M}(\varepsilon).$$

Daraus ergibt sich, wenn der Mittelwert $V_1(P_0)$ existiert, für $\varepsilon\to 0$

$$\lim_{p_0\to P_0} v(p_0)=\lim_{\varepsilon\to 0}\overline{m}(\varepsilon)=\lim_{\varepsilon\to 0}\overline{M}(\varepsilon)=V_1(P_0),$$

was dem aufgestellten Theorem entspricht.

Zu einem allgemeineren Konvergenzsatz gelangen wir mit Hilfe der folgenden Definition: Sei P_0 der Mittelpunkt einer Kalotte mit sphärischem Radius t. Der Flächeninhalt der Kalotte ist $2\pi\ (1-\cos t)$. Wenn das Integral

$$V_2(P_0\,;\,t)=\frac{1}{2\pi(1-\cos t)}\int_{\omega(P_0,\,P')\leqq t} V(P')d\sigma_{P'}$$

existiert (was für $t>0$ immer der Fall ist) und für $t\to 0$ einen Grenzwert

$$V_2(P_0)=\lim_{t=0} V_2(P_0\,;\,t)$$

besitzt, so nennen wir $V_2(P_0)$ den *Kalottenmittelwert* von $V(P)$ im Punkte P_0.

Theorem V. Existiert in einem Punkte P_0 von S der Kalottenmittelwert von $V(P)$

$$V_2(P_0)=\lim_{t\to o}\frac{1}{2\pi(1-\cos t)}\int_{\omega(P_0,\,P')\leqq t} V(P')d\sigma_{P'},$$

so strebt das Poissonsche Integral $v(p_0)$ bei radialer Annäherung $p_0\to P_0$ in K gegen diesen Mittelwert.

Zum Beweis greifen wir auf das Integral in Formel (8) zurück. Durch partielle Integration erhalten wir daraus

$$\frac{1}{2}\int_0^\epsilon F(r,t)\{V_1(P_0\,;\,t)\sin t\}dt=\left[\frac{1}{2}F(r,t).\int_0^t V_1(P\,;\,\xi)\sin\xi\,d\,\xi\right]_{t=o}^{t=\epsilon}$$
$$-\frac{1}{2}\int_0^\epsilon\left\{\int_0^t V_1(P_0\,;\,\xi)\sin\xi\,d\,\xi\right\}\frac{dF(r,t)}{dt}\,dt$$
$$=\frac{1}{2}F(r,\varepsilon).\int_0^\epsilon V_1(P_0\,;\,\xi)\sin\xi\,d\,\xi-\frac{1}{2}\int_0^\epsilon\left\{\frac{1}{\sin^2 t}\int_0^t V_1(P_0\,;\,\xi)\sin\xi\,d\,\xi\right\}\times$$
$$\frac{dF(r,t)}{dt}\sin^2 t\,dt. \quad (10)$$

Beachten wir zunächst, dass das erste Glied der rechten Seite für $r\to 1$ verschwindet. Die Berechnung der ersten Ableitung von $F(r,t)$ in bezug auf t zeigt, dass diese Ableitung ihr Zeichen im betrachteten Intervall nicht ändert, somit der Mittelwertsatz sich auf das Integral des zweiten Gliedes anwenden lässt. Der Klammerausdruck

$$\frac{1}{\sin^2 t}\int_0^t V_1(P_0\,;\,\xi)\sin\xi\,d\,\xi$$

lässt sich auf folgende Weise umformen:

$$\sin^2 t=4\,\sin^2\frac{t}{2}\,\cos^2\frac{t}{2}=2(1-\cos t)\cos^2\frac{t}{2},$$

(nach 7) $$\int_0^t V_1(P_0\,;\,\xi)\sin\xi\,d\,\xi=\frac{1}{2\pi}\int_0^t\int_1^{2\pi} V(p')\sin\xi\,d\,\xi\,d\,\varphi\;;$$

also ist

$$\frac{1}{\sin^2 t}\int_0^t V_1(P_0\,;\,\xi)\sin\xi\,d\,\xi = \frac{1}{2\cos^2\frac{t}{2}}\cdot\frac{1}{2\pi(1-\cos t)}\int_{\omega(P_0,P')\leqq t} V(P')d\sigma_{P'}$$

Ist t^* ein mittlerer Wert zwischen 0 und t, so bezeichnet

$$\frac{1}{2\cos^2\frac{t^*}{2}}\cdot\frac{1}{2\pi(1-\cos t^*)}\int_{\omega(P_0,P')\leqq t^*} V(P')\,d\sigma_{P'} = V_2(P_0\,;\,t^*)\frac{1}{2\cos^2\frac{t^*}{2}}$$

einen Mittelwert des letzten Ausdrucks. Führen wir diesen Mittelwert in Formel (10) ein, so erhalten wir für das zweite Glied

$$-\frac{1}{4\cos^2\frac{t^*}{2}}\cdot V_2(P_0\,;\,t^*)\int_0^\epsilon \frac{dF(rt)}{dt}\sin^2 t\,dt. \tag{11}$$

Für das letzte Integral erhalten wir durch partielle Integration

$$\frac{1}{2}\int_0^\epsilon \frac{dF(r,\,t)}{dt}\sin^2 t\;\;dt = \frac{1}{2}\left|F(r,\,t)\sin^2 t\right|_0^\epsilon - \int_0^\epsilon F(r,\,t)\sin t\cos t\,dt.$$

Unter Berücksichtigung von Theorem III und Formel (8) erhalten wir daraus

$$\lim_{p\to P_0}\frac{1}{2}\int\frac{dF(r,\,t)}{dt}\;\sin^2 t\,dt = -2.$$

Nehmen wir nun an, dass der Kalottenmittelwert $V_2(P_0)$ existiere, so erhalten wir aus (11) und (12) unter Berücksichtigung von Theorem I

$$\lim_{p_0\to P} v(p_0) = V_2(P_0),$$

was zu beweisen war.

Der Kalottenwert $V_2(P_0)$ besteht bei integrierbaren Funktionen fast überall, d.h. höchstens mit Ausnahme einer Punktmenge vom Masse Null ([1]). Es gilt also

Theorem VI. Das Poissonsche Integral $v(p)$ strebt bei radialer Annäherung $p\to P$ in K fast überall gegen den Wert $V(P)$.

§ 4. Integralsätze.

Sei S_r eine Kugelfläche mit Zentrum C und Radius $r<1$. Sei $d\sigma_r$ ein Flächenelement von S_r. Zwischen dem Flächenelement $d\sigma$ von S und dem Flächenelement $d\sigma_r$ von S_r besteht die Beziehung

$$d\sigma_r = r^2 d\sigma.$$

([1]) Vgl. H. Lebesgue, Sur l'intégration des fonctions discontinues. Annales de l'École normale supérieure (3) 27 (1910), p. 361–450.

Theorem VII. (1) *Unter der einzigen Voraussetzung, dass* $|V(P)|$ *integrierbar ist, gelten die Formeln*

$$1.\quad \lim_{r\to 1}\int_{S_r} V(P)d\sigma = \int_S V(P)d\sigma_P.$$

$$2.\quad \lim_{r\to 1}\int_{S_r} |v(p)|\ d\sigma_r = \int_S |V(P)|\ d\sigma_P.$$

Desgleichen, wenn $\mathfrak{M}$ *eine beliebige messbare Punktmenge von* S *und* $\mathfrak{M}_r$ *die entsprechende Punktmenge von* S_r *bedeutet, so gelten die Formeln*

$$3.\quad \lim_{r\to 1}\int_{\mathfrak{M}_r} V(P)d\sigma_r = \int_{\mathfrak{M}_r} V(P)d\sigma_P.$$

$$4.\quad \lim_{r\to 1}\int_{\mathfrak{M}_r} |\ v(p)\ |\ d\sigma_r = \int_{\mathfrak{M}} |\ V(P)\ |\ d\sigma_P.$$

Beweis des ersten Teils von Theorem VII. (Formel 1. und 2.) Jede Fläche S_r liegt ganz im Innern des Bereichs der harmonischen Funktion $v(p)$. Nach dem Mittelwertsatz der Potentialtheorie (2) ist deshalb der Wert von $v(p)$ im Mittelpunkt C von S_r gegeben durch

$$v(C) = \frac{1}{4\pi r^2}\int_{S_r} v(p)d\sigma_r.$$

Anderseits ist aber der Wert $v(C)$ gegeben durch das Poissonsche Integral

$$v(C) = \frac{1}{4\pi}\int_S V(P')\frac{1-\overline{CC}^2}{\overline{CP'}^3}d\sigma_{P'} = \frac{1}{4\pi}\int_S V(P')d\sigma_{P'}.$$

Daraus folgt, weil $d\sigma_r = r^2 d\sigma$,

$$\frac{1}{r^2}\int_{S_r} v(p)d\sigma_r = \int_S v(p)d\sigma = \int_S V(P')d\sigma_{P'}.$$

Weil der letzte Ausdruck von r unabhängig ist, so erhalten wir für $r\to 1$ die erste Formel des Theorems:

$$\lim_{r\to 1}\int_{S_r} v(p)d\sigma_r = \int_{S_r} V(P)d\sigma_P. \tag{12}$$

Ist $V(P)>0$, so ist, weil $F(r,\omega)>0$, auch $v(p)\geqq 0$, und wir erhalten in diesem Fall unmittelbar auch die zweite Formel des Theorems

(1) Einen ähnlichen Satz für die Ebene hat M. Plancherel in der oben zitierten Arbeit (Fussnote 5) ohne Beweis angegeben. Unter der Voraussetzung, dass $V(P)$ auf S beschränkt und daselbst mit Ausnahme endlich vieler Punkte stetig ist, hat O. D. Kellogg einen Beweis gegeben. (Vgl. oben, Fussnote 5).

(2) Vgl. Picard, Traité d'analyse 1, p. 159.

$$\lim_{r\to 1}\int_{S_r}|v(p)|\,d\sigma_r=\int_S|V(P)|\,d\sigma_P. \tag{13}$$

Um diese Formel für den allgemeinen Fall $V(P)\gtreqless 0$ zu beweisen, machen wir folgende Zerlegung:

$$V(P)=V_1-V_2,\ \text{wo}\begin{cases} V=V_1,\ \text{wenn}\ V\geqq 0\\ V=V_2,\ \text{wenn}\ V<0\,.\end{cases}$$

Es ist dann Formel (13) anwendbar auf

$$v_1(p)=\frac{1}{4\pi}\int_S V_1(P')F(r,\,\omega)d\sigma_{P'}$$

und auf

$$v_2(p)=\frac{1}{4\pi}\int_S V_2(P')F(r,\,\omega)d\sigma_{P'}\,.$$

Weil ferner $|V|=|V_1|+|V_2|;\quad |v|=|v_1|+|v_2|$,

so gilt Formel (13) auch für diesen allgemeinen Fall.

Beweis des zweiten Teiles von Theorem VII. (Formel 3. und 4.) Die Formeln 3. und 4. des Theorems ergeben sich aus folgenden zwei Hilfssätzen:

Erster Hilfssatz. Wenn eine Funktion $u(p)$ im Innern der Kugel S definiert ist und bei radialer Annäherung $p\to P$ fast überall, d.h. höchstens mit Ausnahme einer Nullmenge von Punkten, Werte der Randfunktion $U(P)$ zustrebt, wenn ferner zwei beliebige Zahlen δ und ε gegeben sind, so kann man eine Punktmenge E auf S und eine Zahl $R<1$ so bestimmen, dass

$$\left.\begin{aligned} &m(E)>m(S)-\delta\\ &\textit{und dass in jedem Punkte } P \textit{ von } E \textit{ gilt}\\ &|u(p)-U(p)|<\varepsilon \textit{ für } R\leqq r<1\,.\end{aligned}\right\} \tag{14}$$

Beweis dieses Hilfsatzes. Sei E die Menge der P, für welche gilt $u(p)\to U(P)$ für $p\to P$. Sei E' die Komplementärmenge von E. Nach Voraussetzung gilt

$$m(E')=0\,;\ m(E)=m(S)\,.$$

Sei $P=(1,\,\vartheta,\,\varphi)$ ein Punkt von E. Wir bestimmen ein $\rho=\rho\,(\vartheta,\,\varphi)$ so, dass in diesem Punkte

$$|u(p)-U(P)|<\varepsilon \text{ für } \rho\leqq r<1\,.$$

Die Menge E_{ρ_1} enthält alle Punkte von E, für welche $\rho\leqq\rho_1$. Ist $\rho_2>\rho_1$ so ist E_{ρ_1} in E_{ρ_2} enthalten. Haben wir also die Folge

$$\rho_1<\rho_2<\rho_3\ldots<\rho_n\ldots\qquad \lim_{n\to\infty}\rho_n=1\,,$$

so gilt $$E_{\rho_1}(E_{\rho_2}(E_{\rho_3}\ldots(E_{\rho_n}\ldots$$
Jeder Punkt P von E gehört zu einer Menge $E_{\rho i}$ und folglich zu allen Mengen E_{ρ_i+p} $(p=1, 2, 3\ldots)$. Demnach ist, wenn $E_i\equiv E_{\rho_i}$,

$$E=E_1+(E_2-E_1)+(E_3-E_2)+\ldots$$

und $$m(E)=m(E_1)+m(E_2-E_1)+m(E_3-E_2)+\ldots$$

Nun ist aber $m(E_2-E_1)=m(E_2)-m(E_1)$, weil $E_1(E_2$: folglich

$$m(E)=m(E_1)+\left[m(E_2)-m(E_1)\right]+\ldots=\lim_{n\to\infty} m(E).$$

Da nun $m(E_n)\leqq m(S)$ und $m(E)=m(S)$, so ist

$$m(E_n)\to m(S) \text{ für } n\to\infty .$$

Man kann also einen Index N so bestimmen, dass

$$m(S)-m(E_n)<\delta \text{ für } n\geqq N$$

oder $$m(E_n)>m(S)-\delta .$$

Es genügt nun, um die zu beweisende Formel zu erhalten, für E irgendeine der Mengen E_n zu nehmen, für welche $n\geqq N$ ist. Nehmen wir z.B. $E=E_N$, so ist $R=\rho_N$ zu nehmen, und es gilt dann

$$m(E)>m(S)-\delta ,$$

und für jeden Punkt von E

$$|u(p)-U(P)|<\varepsilon \text{ für } R\leqq r<1 .$$

Zweiter Hilfssatz. Wenn zwischen einer Funktion $u(p)$ im Innern der Kugel S und der integrierbaren Randfunktion $U(P)$ die Beziehungen gelten:

1. $u(p)\to U(P)$ *fast überall für* $p\to P$.

2. $\int_S |u(p)|\, d\sigma\to\int_S |U(P)|\, d\sigma$ *für* $r\to 1$, (1)

wenn ferner $\mathfrak{M}$ eine messbare Punktmenge auf S bedeutet, so gilt

$$\int_{\mathfrak{M}} |u(p)-U(P)|\, d\sigma\to 0 \text{ für } r\to 1 .$$

Beweis. Aus der Integrierbarkeit von $U(P)$ folgt: Wenn eine beliebige Zahl ε gegeben ist, so lässt sich ein $\delta=\delta(\varepsilon)$ so bestimmen, dass für jede beliebige messbare Punktmenge H vom Masse $m(H)<\delta$ gilt

$$\int_H |U(p)|\, d\sigma<\varepsilon \qquad (m(H)<\delta). \tag{15}$$

(1) Das Integral $\int_S |u(p)|\, d\sigma$ ist so zu verstehen, dass in ihm p die Oberfläche Sr durchläuft, während $d\sigma$ das entsprechende Flächenelement im Punkte P auf S ist.

In der ersten Voraussetzung lassen sich ferner nach dem ersten Hilfssatz eine Punktmenge $\mathfrak{E}$ auf S und eine Zahl $R<1$ so bestimmen, dass $m(\mathfrak{E})>m(S)-\delta$, und dass in jedem Punkt von $\mathfrak{E}$ gilt: $|u(p)-U(P)|<\varepsilon$ für $R\leqq r<1$, dass also

$$\int_{\mathfrak{E}}|u(p)-U(P)|\,d\sigma<\varepsilon.\,m(\mathfrak{E})<\varepsilon.\,m(S) \text{ für } R\leqq r<1. \quad (16)$$

Ist nun $\mathfrak{E}'$ die Komplementärmenge von $\mathfrak{E}$, also $m(\mathfrak{E}')<\delta$, so folgt aus (15)

$$\int_{\mathfrak{E}'}|U(p)|\,d\sigma<\varepsilon, \quad (15a)$$

folglich ist

$$\int_{\mathfrak{E}'}|u(p)-U(P)|\,d\sigma\leqq\int_{\mathfrak{E}'}|u(p)|\,d\sigma+\varepsilon. \quad (17)$$

Den Wert des letzten Integrals erhalten wir mit Hilfe der zweiten Voraussetzung, aus welcher sich eine Zahl R' ergibt, derart, dass

$$\int_{\mathfrak{E}'}|u(p)|\,d\sigma+\int_{\mathfrak{E}}|u(p)|\,d\sigma-\int_{\mathfrak{E}'}|U(P)|\,d\sigma$$
$$-\int_{\mathfrak{E}}|U(P)|\,d\sigma=\theta\varepsilon \;\; (|\theta|<1) \text{ für } R'\leqq r<1.$$

Unter Beachtung von (15a) und (16) folgt daraus, wenn $\mathfrak{R}$ die grössere der beiden Zahlen R und R' ist

$$\int_{\mathfrak{E}'}|u(p)|\,d\sigma=\theta\varepsilon+\theta'\varepsilon+\theta''\varepsilon.\,m(S) \text{ für } \mathfrak{R}\leqq r<1$$
$$(|\theta|,\ |\theta'|,\ |\theta''|<1) \quad (18)$$

Der Vergleich der Formeln (16), (17) und (18) liefert nun, weil $\mathfrak{E}+\mathfrak{E}'=S$,

$$\int_S|u(p)-U(P)|\,d\sigma\leqq k\varepsilon \text{ für } \mathfrak{R}\leqq r<1, \quad (19)$$

wo $k=[3+2m(S)]$ ist. Weil $\mathfrak{M}<S$ und ε beliebig klein genommen werden kann, so folgt daraus unmittelbar die zu beweisende Formel

$$\int_{\mathfrak{M}}|u(p)-U(P)|\,d\sigma\to 0 \text{ für } r\to 1.$$

Folgerungen aus dem zweiten Hilfssatz. Behalten wir die Voraussetzungen des zweiten Hilfssatzes bei, so ergibt sich aus der letzten Formel:

Erste Folgerung: $$\int_{\mathfrak{M}}|u(p)|\,d\sigma\to\int_{\mathfrak{M}}|U(P)|\,d\sigma \text{ für } r\to 1.$$

Zweite Folgerung: $$\int_{\mathfrak{M}} u(p)\, d\sigma \to \int_{\mathfrak{M}} U(P)\, d\sigma \qquad \text{für } r \to 1\,.$$

Ist ferner $\mathfrak{M}_r$ die Punktmenge auf S_r, welche der Punktmenge $\mathfrak{M}$ auf S entspricht, so folgt, weil $d\sigma_r = r^2 d\sigma$, aus den beiden letzten Beziehungen weiter

Dritte Folgerung: $$\int_{\mathfrak{M}_r} |\, u(p)\,|\; d\sigma_r \to \int_{\mathfrak{M}} |\, U(P)\,|\; d\sigma \text{ für } r \to 1\,.$$

Vierte Folgerung: $$\int_{\mathfrak{M}_r} u(p)\, d\sigma_r \to \int_{\mathfrak{M}} U(P)\, d\sigma \qquad \text{für } r \to 1\,.$$

Anwendung der Hilfssätze auf das Poissonsche Integral. Nach Theorem VI und der bereits bewiesenen zweiten Formel von Theorem VII erfüllt das Poissonsche Integral die Voraussetzungen des ersten und zweiten Hilfssatzes. Die beiden Hilfssätze und die daraus sich ergebenden Folgerungen sind also auf $v(p)$ anwendbar. Die dritte und vierte Folgerung entsprechen dann der dritten und vierten Formel von Theorem VII, welche damit bewiesen sind.

Zweites Kapitel.

Unitätsbedingungen für die Lösung des Dirichlet'schen Problems.

Wenn auf der Kugelfläche S eine *stetige* Funktion $V(p)$ gegeben ist, so existiert eine und nur eine Potentialfunktion $v(p)$, welche im Innern der Kugel regulär ist und bei radialer Annäherung an einen Punkt p von S überall dem Werte $V(p)$ zustrebt. Diese Potentialfunktion ist gegeben durch das Poissonsche Integral([1]). Wird aber zugelassen, dass an einzelnen Punkten $v(p)$ nicht gegen $V(p)$ strebt, so gilt dieser Unitätssatz nicht mehr. Sei z. B. eine Kugel durch folgende Gleichung gegeben

$$a(x^2+y^2+z^2)+x=0.$$

Sei ferner $V(x, y, z)$ eine Potentialfunktion, welche im Innern dieser Kugel regulär ist und bei radialer Annäherung an einen Punkt der Oberfläche überall, mit Ausnahme einer beschränkten Anzahl von Punkten, dem Werte der Randfunktion in diesem Punkte zustrebt; dann genügt denselben Forderungen auch die Funktion $V(x, y, z)+U(x, y, z)$, wo

$$U(x, y, z)=\frac{x+a(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}=\frac{x}{R^3}+\frac{a}{R} \quad (R=x^2+y^2+z^2),$$

([1]) Vgl. Encyclopädie der math. Wiss. 11 C 3 (L. Lichtenstein) p. 218 f.

denn $U(x, y, z)$ ist im Innern der Kugel harmonisch(1) und strebt bei Annäherung an einen Punkt der Kugeloberfläche überall, mit Ausnahme des Ursprungs, gegen Null.

Eine hinreichende Bedingung für die Einzigkeit der Lösung dieses Problems ist in folgendem Unitätssatz enthalten:

Theorem VIII. Sei auf der Kugelfläche S mit Radius 1 eine im Sinne von Lebesgue integrierbare Funktion $V(p)$ gegeben. In diesem Falle gibt es im Innern der Kugel eine, und nur eine Potentialfunktion $v(p)$, welche

1. im ganzen Innern der Kugel regulär, d.h. nebst den partiellen Ableitungen erster und zweiter Ordnung stetig ist;

2. bei radialer Annäherung an einen Punkt P der Oberfläche fast überall, d.h. höchstens mit Ausnahme einer Nullmenge von Punkten dem Werte $V(p)$ zustrebt;

3. die folgende Bedingung erfüllt:

$$\lim_{r\to 1}\int_{S_r} |v(p)|\,d\sigma_r = \int_S |V(P)|\,d\sigma_P,$$

wo S_r eine zu S konzentrische Kugel vom Radius r bedeutet. Diese Potentialfunktion ist durch das Poissonsche Integral gegeben.

Dass das Poissonsche Integral die drei aufgestellten Bedingungen erfüllt, ergibt sich aus den vorausgehenden Untersuchungen. (Vgl. Bemerkungen zu Formel (1), ferner Theorem VI und VII). Es bleibt nur zu beweisen, dass jede andere Potentialfunktion $u(p)$, welche dieselben Bedingungen erfüllt, mit dem Poissonschen Integral $v(p)$ identisch ist.

Seien $v(p_\rho)$ und $u(p_\rho)$ die Werte, welche diese Funktionen in einem Punkte p_ρ der Fläche S_ρ annehmen. Da diese Werte auf S_ρ stetig sind, so lassen sich $v(p)$ und $u(p)$ im Innern von S_ρ immer durch das Poissonsche Integral über S_ρ ausdrücken. Ist also p irgend ein Punkt in K, so wird für $\rho > r$

$$v(p) = \frac{1}{4\pi\rho}\int_{S_\rho} v(p_\rho)\frac{\rho^2 - r^2}{\overline{pp_\rho}^3}\,d\sigma_\rho.$$

(1) $\Delta U = 0$ lässt sich ableiten aus $\Delta\left(\frac{1}{R}\right) = 0$, denn $\Delta U = \Delta\left(\frac{x}{R^3}\right) + a\Delta\left(\frac{1}{R}\right)$; $\frac{x}{R^2} = -\frac{\partial}{\partial x}\left(\frac{1}{R}\right)$, also $\Delta U = -\frac{\partial}{\partial x}\Delta\left(\frac{1}{R}\right) + a\Delta\left(\frac{1}{R}\right) = 0$. — Das angeführte Beispiel entspricht dem von E. Picard für die Ebene gegebenen. (Traité d'analyse II. p. 49.)

$$u(p) = \frac{1}{4\pi\rho}\int\limits_{S_\rho} u(p_\rho)\frac{\rho^2-r^2}{\overline{pp_\rho}^3}\,d\sigma_\rho.$$

$$v(p)-u(p) = \frac{1}{4\pi\rho}\int\limits_{S_\rho}\frac{\rho^2-r^2}{\overline{pp_\rho}^3}\Big[v(p_\rho)-u(p_\rho)\Big]\,d\sigma_\rho.$$

Die linke Seite dieser Gleichung ist unabhängig von ρ; demnach besteht noch der Limes für $\rho\to 1$ und

$$v(p)-u(p) = \lim_{\rho\to 1}\frac{1}{4\pi\rho}\int\limits_{S_\rho}\frac{\rho^2-r^2}{\overline{pp_\rho}^3}\Big[v(p_\rho)-u(p_\rho)\Big]d\sigma_\rho.$$

Die Funktion $\frac{\rho^2-r^2}{\overline{pp_\rho}^3}$ bleibt bei festem p beschränkt und strebt gegen $\frac{1-r^2}{\overline{pP}'^3}$. Es gibt deshalb eine positive konstante Grösse M, derart, dass für alle Punkte p_ρ, für welche $r+\frac{1-r}{2}\leqq\rho\leqq 1$ ist, gilt

$$\frac{1}{4\pi\rho}\;\frac{\rho^2-r^2}{\overline{pp_\rho}^3}\leqq M$$

und folglich

$$\left|\lim_{\rho\to 1}\frac{1}{4\pi\rho}\int\limits_{S_\rho}\frac{\rho^2-r^2}{\overline{pp_\rho}^3}\Big[v(p_\rho)-u(p_\rho)\Big]d\sigma_\rho\right|\leqq M\;\lim_{\rho\to 1}\int\limits_{S_\rho}\Big|v(p_\rho)-u(p_\rho)\Big|\,d\sigma_\rho.$$

Das letzte Integral strebt für $\rho\to 1$ gegen Null; denn sowohl $v(p_\rho)$ als auch $u(p_\rho)$ erfüllen infolge der Bedingungen 2 und 3 die Voraussetzungen des zweiten Hilfssatzes im ersten Kapitel (Seite 50), woraus folgt

$$\int\limits_{S_\rho}\Big|v(p_\rho)-V(P)\Big|\,d\sigma_\rho\to 0 \text{ und} \int\limits_{S_\rho}\Big|u(p_\rho)-V(P)\Big|\,d\sigma_\rho\to 0 \text{ für } \rho\to 1.$$

Nun ist

$$\int\limits_{S_\rho}|v(p_\rho)-u(p_\rho)|\,d\sigma_\rho = \int\limits_{S_\rho}\Big|\Big[v(p_\rho)-V(P)\Big]-\Big[u(p_\rho)-V(P)\Big]\Big|\,d\sigma_\rho$$

$$\leqq\int\limits_{S_\rho}\{|v(p_\rho)-V(P)|+|u(p_\rho)-V(P)|\}\,d\sigma_\rho\to 0$$

$$\text{für } \rho\to 1.$$

Daraus ergibt sich $v(p)\equiv u(p)$, was zu beweisen war.

Drittes Kapitel.

Verhalten des n^{ten} Differentials des Poissonschen Integrals $v(p)$ bei Annäherung des Punktes p an einen Punkt P_0 einer Kalotte, auf welcher die Randfunktion $V(P)$ analytisch ist.

§ 1. Allgemeines über Derivierte und Differentiale. Transformation des Poissonschen Integrals.

Bisher haben wir die Randfunktion $V(P)$ als eine Funktion der Polarkoordinaten ϑ, φ, und das Poissonsche Integral als eine Funktion derselben Polarkoordinaten und des Parameters r betrachtet. Seien nun

$$\vartheta=\vartheta(u, w)\,;\ \varphi=\varphi(u, w) \tag{20}$$

eindeutige, umkehrbare und in einem abgeschlossenen Bereiche $a\leqq u\leqq b$; $c\leqq w\leqq d$ stetige Funktionen der *beliebigen Parameter* u und w. Es ist dann

$$\begin{aligned} v(p)&=v(r,\ \vartheta,\ \varphi)=v(r,\ u,\ w), \\ V(P)&=V(\vartheta,\ \varphi)\ \ =v(u,\ w). \end{aligned} \tag{21}$$

Wenn ferner die Funktionen (20) und ihre Umkehrungen

$$u=u(\vartheta,\ \varphi)\,;\ w=w(\vartheta,\ \varphi)$$

im betrachteten Bereich überall stetige Ableitungen bis zur Ordnung $h+k$ einschliesslich besitzen, so lassen sich die partiellen Ableitungen von $V(P)$ und $v(p)$ von der Ordnung $h+k$ in bezug auf u und w darstellen durch ein Polynom, in welchem ausser numerischen Faktoren nur die Ableitungen von $v(p)$ bezw. $V(P)$ in bezug auf ϑ und φ und die Ableitungen von ϑ und φ in bezug auf u und w bis zur Ordnung $h+k$ vorkommen. Auf analoge Weise lassen sich umgekehrt die Ableitungen von $V(P)$ und $v(p)$ in bezug auf ϑ und φ durch die Ableitungen von $V(P)$ bezw. $v(p)$ in bezug auf u und w und die Ableitungen von u und w in bezug auf ϑ und φ darstellen. Daraus folgt:

Wenn

$$\lim_{p\to P_0}\frac{\partial^{h+k}v(p)}{\partial\vartheta^h\partial\varphi^k}=\frac{\partial^{h+k}V(P_0)}{\partial\vartheta^h\partial\varphi^k},$$

so gilt auch

$$\lim_{p\to P_0}\frac{\partial^{h+k}v(p)}{\partial u^h\partial w^k}=\frac{\partial^{h+k}V(P_0)}{\partial u^h\partial w^k}.$$

Ebenso gilt der umgekehrte Satz.

Um das Verhalten der Ableitungen des Poissonschen Integrals bei

Annäherung an einen Punkt der Oberfläche zu bestimmen, genügt es also, die Ableitungen in bezug auf beliebige Parameter zu untersuchen. Die gefundenen Konvergenzsätze lassen sich unter den gemachten Voraussetzungen immer auf Ableitungen in bezug auf andere Parameter ausdehnen.

Eine besonders geeignete Parameterdarstellung bietet die *stereographische Projektion* der Kugelfläche S auf die Tangentialebene im Punkte P_0 $(1, \vartheta_0, \varphi_0)$. Der Einfachheit wegen nehmen wir an, dass P_0 dem Werte $\vartheta_0 = 0$ entspreche. In der Tangentialebene im Punkte P_0 legen wir ferner ein rechtwinkliges Koordinatensystem $(P_0; x, y)$ mit P_0 als Ursprung, derart dass die positive x-Achse die Schnittlinie der Projektionsebene mit der Halbebene $\varphi = 0$ sei. $\mathfrak{P}$ (x, y) bedeute die Projektion des Punktes $P(1, \vartheta, \varphi)$ und auf gleiche Weise $\mathfrak{P}'(x', y')$ diejenige des Punktes $P'(1, \vartheta,' \varphi')$. Führen wir ferner die Bezeichnungen

$$\overline{\mathfrak{P}p_0} = \rho\,; \quad \overline{\mathfrak{P}'p_0} = \rho'$$

ein, so sind (ρ, φ), $(\rho,' \varphi')$ nichts anderes als die Polarkoordinaten von $\mathfrak{P}$, $\mathfrak{P}'$ in der Projektionsebene. Es gelten dann die Formeln

$$\rho^2 = x^2 + y^2\,; \quad \cos\,\varphi = \frac{x}{\rho}\,; \quad \sin\,\varphi = \frac{y}{\rho},$$

$$\rho'^2 = x'^2 + y'^2\,; \quad \cos\,\varphi' = \frac{x'}{\rho'}\,; \quad \sin\varphi' = \frac{y'}{\rho'},$$

$$\rho = 2\operatorname{tg}\frac{\vartheta}{2}; \quad \rho' = 2\operatorname{tg}\frac{\vartheta'}{2}.$$

Daraus leiten sich die folgenden Formeln leicht ab:

$$\sin\vartheta = \frac{4\rho}{4+\rho^2}\,; \; \cos\vartheta = \frac{4-\rho^2}{4+\rho^2}\,; \; \sin\vartheta' = \frac{4\rho'}{4+\rho'^2}\,; \; \cos\vartheta' = \frac{4-\rho'^2}{4+\rho'^2}, \tag{23}$$

und

$$\begin{aligned} \cos\omega &= \cos\vartheta\,\cos\vartheta' + \sin\vartheta\,\sin\vartheta'\,\cos(\varphi-\varphi') \\ &= \frac{(4-\rho^2)(4-\rho'^2) + 16(xx'+yy')}{(4+\rho^2)\,(4+\rho'^2)}. \end{aligned} \tag{24}$$

Eine Bemerkung ist noch zu machen über die Beziehungen zwischen den *partiellen Ableitungen und den Differentialen.* Es ist klar, dass ein Konvergenzsatz, welcher für alle partiellen Ableitungen einer bestimmten Ordnung gilt, auch für das Differential derselben Ordnung seine Gültigkeit bewahrt, und umgekehrt.

§ 2. Konvergenzsatz.

Um definieren zu können, was wir unter einer analytischen Funktion des Punktes P auf der Kugelfläche S verstehen, nehmen wir einen beliebigen Punkt P_0 an, doch so, dass er mit dem Gegenpol von P nicht zusammenfällt und projizieren S stereographisch auf die Tangentialebene in P_0. Dadurch wird $V(P)=v(x, y)$ eine Funktion des Projektionspunktes $\mathfrak{P}(x, y)$ von P. Definitionsweise ist dann $V(P)$ *im Punkte P analytisch*, wenn $v(x, y)$ eine im Punkte (x, y) analytische Funktion der reellen Variablen x, y ist, d.h. wenn in der Umgebung des Punktes (x, y) sich $v(x', y')$ darstellen lässt durch die konvergente Taylorsche Reihe

$$v(x', y')=v(x, y)+\sum_{n=1}^{\infty}\frac{1}{n!}\left[(x'-x)\frac{\partial}{\partial x}+(y'-y)\frac{\partial}{\partial y}\right]^{(n)}v(x, y).$$

$V(P)$ heisst dann analytisch in einem Bereiche Ω der Kugelfläche, wenn sie in jedem inneren Punkte P von Ω analytisch ist.

Man sieht leicht, dass die gegebene Definition unabhängig ist von der Lage der Projektionsebene.

Theorem IX. Wenn die Randfunktion $V(P)$ im Bereiche Ω analytisch ist, so strebt das n^{te} Differential $d^n_{x,y}v(p)$ des Poissonschen Integrals bei beliebiger Annäherung $p\to P_0$ in K an einen Punkt P_0 des Bereiches Ω gegen $d^n_{xy}V(P_0)$. Die Annäherung ist gleichmässig in P_0, wenn P_0 im Innern des Bereiches liegt.

Dieses Theorem stützt sich auf die Tatsache, dass das Poissonsche Integral unter der gemachten Voraussetzung auf der Aussenseite des Bereiches Ω eine *analytische Fortsetzung* besitzt, wie im folgenden § gezeigt wird. Daraus folgt, dass Ableitungen von $v(p)$ auf beiden Seiten des Bereiches Ω stetig sind und beim Durchgang des Punktes p durch die Kugelfläche in P_0 stetig in die Ableitungen der hier analytischen Funktion $V(P)$ übergehen.

§ 3. Die analytische Fortsetzung des Poissonschen Integrals ausserhalb der Kugel.

Hilfssatz I. Es sei $h(p)$ eine reguläre Potentialfunktion innerhalb K. Wenn dann für jeden inneren Punkt P eines Bereiches Ω der Kugeloberfläche der Limes $\mathrm{Lim}\ h(p)=H(P)$ existiert und eine analytische Funktion des Punktes P im Bereiche Ω darstellt, so lässt sich $h(p)$ auf der Aussenseite von Ω analytisch als reguläre Potentialfunktion fortsetzen.

Beim Beweis können wir uns auf den Fall beschränken, wo Ω eine Kalotte ist.—Der Hilfssatz lässt sich durch eine einfache Transfor-

mation auf folgendes Analogon eines Schwarz'schen Satzes [1] zurückführen:

Schwarz'scher Satz. Die Funktión $u(p)=u(x, y, z)$ sei im Innern der Halbkugel

$$x^2+y^2+z^2<R^2, \quad z>0$$

eine reguläre Potentialfunktion. Ist dann

$$u(x, y, 0)=\lim_{z\to 0} u(x, y, z)$$

eine im Gebiete $x^2+y^2<R^2$ analytische Funktion der beiden reellen Variablen x, y, so lässt sich $u(x, y, z)$ auf der unteren Seite $z<0$ der Kreisfläche $x^2+y^2<R^2$ analytisch fortsetzen.

Bevor wir diesen Satz beweisen, führen wir die oben angedeutete Transformation durch. Sei P_0 der Mittelpunkt der Kalotte Ω; x, y, z rechtwinklige Koordinaten mit dem Gegenpol von P_0 als Ursprung und mit $\overrightarrow{CP_0}$ als Richtung der positiven z–Achse. Bei der Transformation durch reziproke Radien

$$\xi=\frac{4x}{x^2+y^2+z^2}, \quad \eta=\frac{4y}{x^2+y^2+z^2}, \quad \zeta=\frac{4z}{x^2+y^2+z^2}$$

entspricht jedem Punkte (x, y, z) des Kugelraumes ein Punkt (ξ, η, ζ) des Halbraumes $\zeta>2$, und jedem Punkt der Kugelfläche ein Punkt der Ebene $\zeta=2$. Der Kalotte Ω entspricht eine Kreisfläche Ω'. Es bezeichne nun P einen Punkt in Ω und (ξ, η) den entsprechenden Punkt in Ω'. Es sei ferner

$$\mathfrak{H}(\xi, \eta)=H(P) \text{ und } \mathfrak{h}(\xi, \eta, \zeta)=h(p),$$

also

$$\mathfrak{h}(\xi, \eta, 2)=\mathfrak{H}(\xi, \eta) \text{ in } \Omega'.$$

Nach einem Satz von William Thomson [2] wird die Funktion

$$\frac{\mathfrak{h}(\xi, \eta, \zeta)}{\sqrt{\xi^2+\eta^2+\zeta^2}}=\frac{h(p)\sqrt{x^2+y^2+z^2}}{4}$$

eine reguläre Potentialfunktion von ξ, η, ζ im Innern des Halbraumes $\zeta>2$. Diese Funktion nimmt in Ω' die Randwerte

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(1) Der Schwarz'sche Satz findet sich für die Ebene bewiesen z. B. bei Osgood, Funktionentheorie, Leipzig (1920) I, p. 665.—Vgl. Picard, Traité d'analyse II. p. 298.

(2) Vgl. Picard, Traité d'analyse II. p. 64. Der Satz lautet: Wenn eine Funktion $V(x, y, z)$ harmonisch ist, so ist es auch

$$\frac{1}{\sqrt{x^2+y^2+z^2}}V\left(\frac{k^2x}{x^2+y^2+z^2}, \frac{k^2y}{x^2+y^2+z^2}, \frac{k^2z}{x^2+y^2+z^2}\right).$$

$$\frac{\mathfrak{H}(\xi,\ \eta)}{\sqrt{\xi^2+\eta^2+4}}$$

an. Nun ist wegen den Voraussetzungen $\mathfrak{H}(\xi,\eta)$ analytisch in ξ,η. Dasselbe gilt also von $\frac{\mathfrak{H}(\xi,\ \eta)}{\sqrt{\xi^2+\eta^2+4}}$. Die Voraussetzungen des Schwarzschen Satzes sind also erfüllt. Folglich lässt sich $\frac{\mathfrak{h}(\xi,\ \eta,\ \zeta)}{\sqrt{\xi^2+\eta^2+\zeta^2}}$, mithin auch $\mathfrak{h}(\xi,\ \eta,\ \zeta)$, auf der Seite $\zeta<2$ der Kreisfläche Ω' analytisch fortsetzen. Damit ist gezeigt, dass $h(p)$ auf der Aussenseite der Kalotte Ω eine analytische Fortsetzung hat.

Um *den Schwarz'schen Satz* zu beweisen, nehmen wir zuerst den Fall, wo $u(x,\ y,\ 0)\equiv 0$ in $x^2+y^2<R^2$. Ist dann $0<R_0<R$ und bedeuten $(r,\ \vartheta,\ \varphi)$ die Polarkoordinaten von $(x,\ y,\ z)$, so konstruieren wir durch Spiegelung auf die Oberfläche der unteren Halbkugel vom Radius R_0 die Funktion

$$h(R_0,\ \vartheta,\ \varphi)=-h(R_0,\ \pi-\vartheta,\ \varphi) \qquad 0\leqq\vartheta\leqq\frac{\pi}{2},\ 0\leqq\varphi\leqq 2\pi \tag{22}$$

und bilden das Poissonsche Integral

$$U(r,\vartheta,\varphi)=\frac{1}{4\pi R_0}\int_0^{\pi}\int_0^{2\pi}h(R_0,\ \vartheta',\ \varphi')\frac{R_0^2-r^2}{(R_0^2-2rR_0\cos\omega+r^2)^{3/2}}\sin\vartheta'\,d\vartheta' d\varphi'.$$

Dieses Integral definiert eine innerhalb der Kugel vom Radius R reguläre Potentialfunktion, die wegen (22) auf der Äquatorebene $\vartheta=\frac{\pi}{2}$ verschwindet. Ferner ist wegen Theorem III:

$$\lim_{r\longrightarrow R_0} U(r,\ \vartheta,\ \varphi)=h(R_0,\ \vartheta,\ \varphi).$$

Folglich haben U und u dieselben stetigen Werte auf der geschlossenen Fläche, die aus der Äquatorebene und der oberen Halbkugelfläche gebildet ist. Nach einem bekannten Satze sind sie also einander gleich in dem von dieser Fläche eingeschlossenen Raum. Nun existiert aber U auch in der unteren Halbkugel als reguläre Potentialfunktion. Folglich hat auch u daselbst eine analytische Fortsetzung.

Der *allgemeine Fall*, wo $u(x,\ y,\ 0)$ in $x^2+y^2<R^2$ eine beliebige analytische Funktion ist, lässt sich auf den soeben behandelten Fall zurückführen. Nach dem Existenzsatz von Cauchy-Kowalewsky gibt es eine Funktion $f(x,\ y,\ z)$, welche auf beiden Seiten der Kreisfläche $(x^2+y^2<R^2$,

$z=0$) harmonisch und analytisch ist und sich für $z=0$ auf $u\,(x,\,y,\,0)$ reduziert[1].

Bilden wir im oberen Halbraum die Differenz

$$u(x,\,y,\,z)-f(x,\,y,\,z).$$

Diese Differenz ist im oberen Halbraum harmonisch und verschwindet für $z=0$. Sie hat also nach dem vorigen Fall im unteren Halbraum eine analytische Fortsetzung. Dasselbe gilt für $u(x,\,y,\,z)$ als die Summe von zwei im unteren Halbraum noch regulären Potentialfunktionen.

Damit ist sowohl das Schwarz'sche Lemma als auch Hilfssatz I bewiesen. Die Anwendung des letzteren auf das Poissonsche Integral $v(p)$ stützt sich auf die Tatsache, dass $v(p)$ im Innern der Kugel eine reguläre Potentialfunktion ist und (nach Theorem III) bei Annäherung an einen Stetigkeitspunkt P_0 der Randfunktion $V(P)$ dem Werte $V(P_0)$ zustrebt.

Viertes Kapitel.

Verhalten des n^{ten} Differentials des Poissonschen Integrals $v(p)$ bei Annäherung des Punktes p an einen Punkt p_0 der Oberfläche S, in welchem $V(P)$ ein Differential n^{ter} Ordnung besitzt.

§ 1. Allgemeiner Satz über das Verhalten eines Differentials von $v(p)$ für $p \to P_0$.

Bevor wir den eigentlichen Gegenstand dieses Kapitels behandeln können, müssen wir folgenden, das Theorem I auf die Differentiale einer beliebigen Ordnung ausdehnenden Satz vorausschicken.

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(1) Eine solche Funktion ist z. B. wenn (x_0, y_0) ein Punkt der Kreisfläche $(x^2+y^2<R^2,\ z=0)$ ist:

$$f(x,y,z)=u(x_0,y_0,0)+\sum_n\sum_{ik\lambda}\alpha_{ik\lambda}(x-x_0)^i(y-y_0)^k z^{2\lambda}\frac{\partial^{i+k}}{\partial x^i\partial y^k}\Big[(-1)^\lambda \overset{(\lambda)}{\Delta} u(x_0,y_0,0)\Big],\ \text{wo } n=1,2\,..\infty;$$

$$i,\ k=0,1,2\,..\,n;\ \lambda=0,1,2\ldots\begin{cases}\dfrac{n}{2} & \text{für } n \text{ gerade}\\[4pt] \dfrac{n-1}{2} & \text{für } n \text{ ungerade}\end{cases};\ i+k+2\lambda=n;$$

$$\alpha_{ik\lambda}=\frac{1}{i!\,k!\,(2\lambda)!};\ \overset{(\lambda)}{\Delta}u=\Delta\big(\overset{(\lambda-1)}{\Delta}u\big);\ \Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

Es lässt sich unabhängig vom Satze von Cauchy-Kowalewska zeigen, dass diese Funktion $f(x,\,y,\,z)$ auf beiden Seiten von $z=0$ analytisch und harmonisch ist und sich für $z=0$ auf $u(x,\,y,\,0)$ reduziert.

Theorem X. Das Verhalten des n^{ten} Differentials in bezug auf x, y (stereographische Projektion) des Poissonschen Integrals $v(p)$ bei beliebiger Annäherung $p \rightarrow P_0$ in K hängt nur von den Werten der Randfunktion $V(P)$ in der Umgebung von P_0 ab.

Nach den Regeln über die Differentiierung unter dem Integralzeichen lässt sich das n^{te} Differential von $v(p)$ durch folgende Gleichung ausdrücken:

$$d^n v(p) = \frac{1}{4\pi} \int_S V(P')\ d^n F(r, \omega)\ d\sigma_{P'}.$$

Mit Hilfe dieser Formel lässt sich Theorem X schärfer fassen: *Auf der ganzen Kugel gelten gleichmässig in P_0 folgende Grenzwerte:*

$$\lim_{p \rightarrow P_0} \frac{1}{4\pi} \int_{\omega(P_0, P') > \varepsilon} V(P')\ d^n F(r, \omega)\ d\sigma_{P'} = 0 \tag{25 a}$$

oder

$$\lim_{p \rightarrow P_0} \left[d^n v(p) - \frac{1}{4\pi} \int_{\omega(P_0, P') \leqq \varepsilon} V(P)\ d^n F\ (r, \omega)\, d\sigma_P \right] = 0. \tag{25 b}$$

Ist ferner $\mathfrak{M}$ irgend eine von P_0 abhängige, messbare Menge von Punkten P', welche der Bedingung $\omega\ (P_0, P') > \varepsilon$ genügen, so gilt noch gleichmässig in P_0

$$\lim_{p \rightarrow P_0} \frac{1}{4\pi} \int_{\mathfrak{M}} V(P')\ d^n F(r, \omega)\, d\sigma_{P'} = 0. \tag{25 c}$$

Ist nämlich ε eine beliebige positive Grösse $< \pi$, so lässt sich schreiben:

$$\left| \frac{1}{4\pi} \int_{\omega(P_0, P') > \varepsilon} V(P')\ d^n F(r, \omega)\, d\sigma_{P'} \right| \leqq \max_{\varepsilon < \omega \leqq \pi} |\, d^n F(r, \omega) \frac{1}{4\pi} \int_S |\, V(P') \,|\, d\sigma_{P'}$$

$$\leqq \frac{M}{4\pi} (1 - r^2) \max_{\varepsilon < \omega \leqq \pi} \left| d^n \left(\frac{1}{\overline{pP'}^3} \right) \right|.$$

Nun ist aber $1/\overline{pP'}^3$ eine analytische Funktion der zwei reellen Variablen r und ω, welche nur für das Wertepaar $\omega = 0$, $r = 1$ eine Singularität hat. Demnach sind die n^{ten} Ableitungen von $1/\overline{pP'}^3$ im betrachteten Bereich regulär. $d^n F(r, \omega) = (1 - r^2)\ d^n \left(\frac{1}{\overline{pP'}^3} \right)$ ist also beschränkt und strebt folglich für $r \rightarrow 1$ gegen Null, und zwar gleichmässig auf der ganzen Kugel.

§ 2. $V(P)$ besitzt im Punkt P ein Differential n^{ter} Ordnung.

Um weiter zu gehen, müssen wir den Begriff des n^{ten} Differentials von $V(P)$ in einem Punkte P_0 in bezug auf die stereographischen Koordinaten (x, y) genau präzisieren.

Seien (x_0, y_0) die stereographischen Koordinaten von P_0; (x, y) diejenigen von P. Setzen wir $V(P) \equiv v(x, y)$ so verstehen wir, wenn wir sagen, $V(P)$ besitze im Punkte P_0 ein n^{ter} Differential, folgendes:

1. Alle partiellen Ableitungen $\dfrac{\partial^{h+k} v(x, y)}{\partial x^h \partial y^k}$ $(h+k \leqq n)$ existieren im Punkte (x_0, y_0).

2. In der Umgebung von (x_0, y_0) gilt die Entwicklung

$$v(x, y) = v(x_0, y_0) + \sum_{\nu=1}^{n} \frac{1}{\nu!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{(\nu)} v(x_0, y_0)$$
$$+ \left[\sqrt{(x-x_0)^2 + (y-y_0)^2}\,\right]^n \Omega(x_0, y_0; x, y).$$

Betrachten wir $\Omega(x_0, y_0; x, y)$ als Funktion von (x, y) im Gebiete $0 \leqq \sqrt{(x-x_0)^2 + (y-y_0)^2} \leqq \eta$, und bezeichnen wir durch $\overline{\Omega}(P_0, \eta)$ die obere Schranke des absoluten Wertes von Ω in diesem Gebiet, so soll ferner gelten

$$\overline{\Omega}(P_0, \eta) \to 0 \quad \text{für } \eta \to 0.$$

Unter $d^n V(P_0) = d^n_{xy} v(x_0, y_0)$ verstehen wir sodann die Form n^{ter} Ordnung in dx, dy:

$$d^n V(P_0) = d^n_{xy} v(x_0, y_0) = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^{(n)} v(x_0, y_0).$$

Wir bezeichnen kurz durch $V_n(P_0, P)$ den Ausdruck

$$V_n(P_0, P) = v(x_0, y_0) + \sum_{\nu=1}^{n} \frac{1}{\nu!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{(\nu)} v(x_0, y_0),$$

sodass

$$V(p) = v(x, y) = V_n(P_0, P) + \left[\sqrt{(x-x_0)^2 + (y-y_0)^2}\right]^n : \Omega(x_0, y_0; x, y). \tag{26}$$

Ferner ist dann

$$d^\nu V_n(P_0, P_0) = d^\nu V(P_0), \quad \nu \leqq n. \tag{27}$$

Theorem XI. Wenn die Randfunktion $V(P)$ im Punkte P_0 ein Differential n^{ter} Ordnung besitzt, so strebt das n^{te} Differential des Poissonschen Integrals $V(P)$ bei Annäherung $p \to P_0$ in K gegen den Wert des n^{ten} Differentials von $V(P)$ im Punkte P_0, vorausgesetzt, dass p innerhalb eines räumlichen Winkels bleibt, dessen Scheitel in P_0 liegt und der die Kugelfläche S nicht berührt.

Zur Vereinfachung des Beweises denken wir uns das Achsensystem

so gelegen, dass P_0 im Pol liege. Es ist dann $(x_0, y_0)=(0, 0)$ zu nehmen. Nach Theorem X können wir bei Annäherung $p \to P_0$ das Integrationsfeld auf die Umgebung ε von P_0 beschränken, in welcher die oben angeführten Bedingungen 2. und 3. erfüllt sind und somit Gleichung (26) gilt. Es ist also, wenn der Limes der rechten Seite der folgenden Gleichung existiert,

$$\lim_{p \to P_0} d^n v(p) = \lim_{p \to P_0} \frac{1}{4\pi} \int_{\omega(P_0, P') \leqq \varepsilon} [V_n(P_0, P') + \rho'^n \Omega(x_0, y_0; x, y)] d^n F(r, \omega) d\sigma_{P'},$$

wo $\rho' = \sqrt{x'^2 + y'^2}$. Ferner ist, weil $V_n(P_0, P')$ eine analytische Funktion von P' auf der ganzen Kalotte ist, nach Theorem IX und Formel (27)

$$\lim_{p \to P_0} \frac{1}{4\pi} \int_{\omega(P_0, P') \leqq \varepsilon} V_n(P_0, P') d^n F(r, \omega) d\sigma_{P'} = d^n V_n(P_0, P_0) = d^n V(P_0).$$

Um das Theorem XI zu beweisen, ist also nur noch zu zeigen, dass

$$\lim_{p \to P_0} \frac{1}{4\pi} \int_{\omega(P_0, P') \leqq \varepsilon} \rho'^n \Omega(x_0, y_0; x, y) d^n F(r, \omega) d\sigma_{P'} = 0, \quad (x_0, y_0) = (0, 0), \tag{28}$$

wenn p innerhalb des im Theorem genannten räumlichen Winkels bleibt. Wir führen den Beweis der letzten Formel zuerst für radiale und dann für nichtradiale Annäherung.

1. *Radiale Annäherung.* Es ist zunächst $d^n F(r, \omega)$ zu berechnen. Mit Hilfe der Substitutionen

$$t = \cos\omega; \qquad u = 2r(1-t)$$

erhalten wir für $F(r, \omega)$

$$F(r, \omega) = \frac{1-r^2}{(1-2r\cos\omega + r^2)^{3/2}} = \frac{1-r^2}{[(1-r)^2 + 2r(1-t)]^{3/2}}$$
$$= \frac{1-r^2}{[(1-r)^2 + u]^{3/2}} = G(r, u). \tag{29}$$

Für $u = 2r(1-t)$ erhalten wir aus Formel (24) nach einigen Vereinfachungen den Wert

$$u = \frac{16r}{4+\rho'^2} \cdot \frac{(x'-x)^2 + (y'-y)^2}{4+\rho^2} = \frac{16r}{4+\rho'^2} \cdot \frac{\delta^2}{4+\rho^2}, \tag{30}$$

in welchem $\delta^2 = (x'-x)^2 + (y'-y)^2$ das Quadrat der Entfernung $\overline{\mathfrak{P}\mathfrak{P}'}$ ($\mathfrak{P}$ und $\mathfrak{P}'$ sind, wie in Seite 56 bemerkt wurde, die stereographischen Bilder der Punkte P und P', und wo $\rho^2 = \overline{P_0\mathfrak{P}}^2$, $\rho'^2 = \overline{P_0\mathfrak{P}'}^2$ ist.—u ist eine analytische, reguläre Funktion von x, y und x', y' im betrachteten Gebiet. Dasselbe gilt also von allen Differentialen $du, d^2u, \ldots d^n u$ von u in bezug auf x, y, welche also beschränkt bleiben.

Um das Differential $d^nF(r, \omega)=d^nG(r, u)$ zu berechnen, lässt sich folgende, von Fàa di Bruno([1]) gegebene Formel anwenden:

$$d^n G(r, u)=\sum_{r,k} \frac{n!}{k_1!\,k_2!\ldots k_n!}\cdot\frac{\partial^r G}{\partial u^r}\left(\frac{du}{1}\right)^{k_1}$$

$$\times\left(\frac{d^2u}{2!}\right)^{k_2}\ldots\left(\frac{d^nu}{n!}\right)^{k_n} \qquad (31)$$

Die Summe umfasst alle ganzen, nicht negativen Werte von $v, k_1 \ldots k_n$, welche folgende Bedingungen erfüllen:

$$r=k_1+k_2+\ldots+k_n,$$

$$n=k_1+2k_2+\ldots+nk_n.$$

Aus den beiden letzten Beziehungen folgt weiter

$$k_1+n-2r=k_3+2k_4+\ldots+(n-2)\,k_n \geqq 0. \qquad (32)$$

Dank dieser wichtigen Ungleichung brauchen wir, wie sich im folgenden zeigen wird, nur das erste Differential von u ausführlich zu berechnen. Aus (30) erhalten wir dafür

$$du=\frac{-32r}{4+\rho'^2}\cdot\frac{[(x'-x)(4+\rho^2)+x\delta^2]dx+[(y'-y)(4+\rho^2)+y\delta^2]dy}{(4+\rho^2)^2}. \quad (33)$$

Da wir jetzt nur radiale Annäherung betrachten, so ist $(x, y)=(0, 0)$ zu nehmen; also

$$(du)_{x=y=0}=\frac{8r}{4+\rho'^2}\Big[x'dx+y'dy\Big]$$

$$=-\frac{8r\rho'}{4+\rho'^2}\Big[\cos\varphi'dx+\sin\varphi'dy\Big]\equiv\rho'\,.\,C(r,P'). \qquad (33a)$$

$C(r, P')$ ist eine für $\rho'\to 0$ beschränkte Funktion im betrachteten Gebiet. Ferner ergibt eine einfache Berechnung aus (29)

$$\frac{\partial^r G}{\partial u^r}=\left(-\frac{1}{2}\right)^r.3.5\ldots\Big[3+2(r-1)\Big]\frac{1-r^2}{r_1^{3+2r}},$$

$$[r_1^2=(1-r)^2+u]. \qquad (34)$$

Fassen wir nun unter $C_{rk}(r, P')$ den Wert $[C(r, P')]^{k_1}$, sowie alle in (31) und (34) vorkommenden konstanten Faktoren und höheren Differentiale zusammen, so erhalten wir für (31) bei radialer Annäherung

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([1]) Fàa di Bruno, Einleitung in die Theorie der binären Formen, Leipzig (1884) p. 3.—Vgl. Franz Mayer, Math. Annalen 30 (1889), p. 454 und 464.

$$[d^n G(r, u)]_{x=y=0} = [d^n F(r, \omega)]_{x=y=0}$$
$$= (1-r^2) \sum_{\gamma, k} C_{\gamma k}(r, P') \frac{\rho'^{k_1}}{r_1^{3+2\gamma}}, \qquad (35)$$

wo $C_{rk}(r, P')$ eine im betrachteten Gebiet beschränkte Funktion ist.

Gehen wir nun auf die zu beweisende Formel (28) zurück. Setzen wir daselbst den in (35) dargestellten Wert für $d^n F(r, \omega)$ ein, so erhalten wir

$$\frac{1}{4\pi}\int_{\omega(P_0,P')\leqq\varepsilon} \rho'^n \Omega d^n F(r,\omega) d\sigma_{P'} = \sum_{r, k} \frac{1-r^2}{4\pi}\int_{\omega(P^0,P')\leqq\varepsilon} C_{\gamma k}(r, P')\Omega . \frac{\rho'^{n+k_1}}{r_1^{3+2r}} d\sigma_{P'};$$
$$(x=y=0) \qquad (36)$$

Für den unter dem Integralzeichen vorkommenden Bruch lässt sich schreiben

$$\frac{\rho'^{n+k_1}}{r_1^{3+2\gamma}} = \frac{1}{r_1^3}\left[\left(\frac{\rho'^2}{r_1^2}\right)^\gamma \rho'^{n+k_1-2\gamma}\right]; \; n+k_1-2\gamma \geqq 0 \quad \text{(nach 32).}$$

Aus $\rho' = 2\mathrm{tg}\,\frac{\vartheta'}{2} = \frac{2\sin\vartheta'}{1+\cos\vartheta'}$ und $r_1^2 = (1-r)^2 + 2r(1-\cos\vartheta')$ folgt, weil wir $\varepsilon < \frac{\pi}{2}$ voraussetzen können,

$$\frac{\rho'^2}{r_1^2} = \frac{4\sin^2\vartheta'}{(1+\cos\vartheta')^2(1-r)^2 + 2r(1-\cos\vartheta')}$$
$$< \frac{4\sin^2\vartheta'}{2r(1+\cos\vartheta')^2(1-\cos\vartheta')} < \frac{2\sin^2\vartheta'}{r\sin^2\vartheta'} = \frac{2}{r}.$$

Nach diesen Abschätzungen gibt es zwei positive, endliche Zahlen M_{rk} und M_n, derart dass

$$M_{rk} > \left| C_{rk}(r, P') \cdot \frac{\rho'^{2r}}{r_1^{2r}} \cdot \rho'^{n+k_1-2r} \right|$$

und

$$M_n = \sum_{r, k} M_{rk}$$

und folglich

$$\left|\frac{1}{4\pi}\int_{\omega(P_0,P')\leqq\varepsilon} \rho'^n \Omega\, d^n F(r, \omega)\, d\sigma_{P'}\right| \leqq M_n \frac{1-r^2}{4\pi}\int_{\omega(P_0,P')\leqq\varepsilon} \frac{|\Omega|}{r_1^3} d\sigma_{P'} \quad (x=y=0).$$

Nach dem Mittelwertsatz und Formel (2) ist aber *für ein beliebiges p*

$$\frac{1-r^2}{4\pi}\int_{\omega(P_0,P')\leqq\epsilon} \frac{|\Omega|}{r_1^3} d\sigma_{P'} \leqq \overline{\Omega}(P_0, \varepsilon) . \frac{1-r^2}{4\pi}\int_s \frac{1}{r_1^3} d\sigma_{P'} = \overline{\Omega}(P_0, \varepsilon) \quad (37)$$

wo $\Omega(P_0, \varepsilon)$, wie in der Erklärung der Voraussetzung (3. Bedingung) gesagt wurde, die obere Schranke der absoluten Werte von Ω auf der Kalotte $\omega(P_0, P') \leqq \varepsilon$ ist und mit $\varepsilon \to 0$ gegen Null strebt. Damit ist für radiale Annäherung bewiesen, dass

$$\lim_{p_0 \to P_0} \left| \frac{1}{4\pi} \int_{\omega(P_0, P') \leqq \varepsilon} \rho'^n \Omega d^n F(r, \omega)\, d\sigma_{P'} \right| < M_n \cdot \overline{\Omega}(P_0, \varepsilon) \to 0 \text{ für } \varepsilon \to 0 .$$

2. *Nichtradiale Annäherung.* Nach Voraussetzung bleibt p innerhalb eines räumlichen Winkels, dessen Scheitel in P_0 liegt, und dessen Seiten die Kugelfläche S nicht berühren. Der grösste Winkel $\sphericalangle pP_0C$ für einen Punkt p innerhalb dieses räumlichen Winkels sei Θ. Es ist also $\Theta < \frac{\pi}{2}$. Um einen passenden Ausdruck für $d^nF(r, \omega)$ zu erhalten, können wir von den im vorigen Fall entwickelten Formeln (20) bis (34) ausgehen. Aus (33) erhalten wir, wenn $x' - x = \delta \cos \Psi$ und $y' - y = \delta \sin \Psi$ gesetzt wird (wo Ψ den Winkel zwischen der x-Achse und der Strecke δ bedeutet),

$$du = \delta \left[\frac{-32r}{4 + \rho'^2} \cdot \frac{[(4+\rho^2) \cos \Psi + x\delta]dx + [(4+\rho^2) \sin \Psi + y\delta]dy}{(4+\rho^2)^2} \right] \equiv \delta .\ C(r, p, P').$$

$C(r, p, P')$ ist eine für das ganze betrachtete Gebiet beschränkte Funktion. Fassen wir wieder unter $C_{rk}(r, p, P')$ die Funktion $[C(r, p, P')]^{k_1}$, sowie alle in (31) und (34) vorkommenden konstanten Faktoren und höheren Differentiale von u zusammen, so ergibt sich für (31)

$$d^n G(r, u) = d^n F(r, \omega) = (1 - r^2) \sum_{r, \kappa} C_{\gamma k}(r, p, P') \cdot \frac{\delta^{k_1}}{r_1^{3+2\gamma}}, \tag{38}$$

wo wiederum $C_{rk}(r, p, P')$ eine beschränkte Funktion ist.

Setzen wir den gefundenen Wert $d^nF(r, \omega)$ in das Integral der Formel (28) ein, so ergibt sich

$$\frac{1}{4\pi} \int_{\omega(P_0, P') \leqq \varepsilon} \rho'^n \Omega d^n F(r, \omega) d\sigma_{P'} = \sum_{r, k} \frac{1 - r^2}{4\pi} \int_{\omega(P_0, P') \leqq \varepsilon} C_{\gamma k}(r, p, P') \Omega \cdot \frac{\rho'^n \delta^{k_1}}{r_1^{3+2\gamma}} d\sigma_{P'}. \tag{39}$$

ρ, ρ' und δ sind die Seiten des Dreiecks $P_0 \mathfrak{P} \mathfrak{P}'$ ($\mathfrak{P}$ und $\mathfrak{P}'$ = stereographische Bilder von P und P'). Es ist also $\rho' \leqq \rho + \delta$; ebenso ist $\delta \leqq \rho + \delta$; demnach gilt

$$\frac{\rho'^n \delta^{k_1}}{r_1^{3+2\gamma}} \leqq \frac{(\rho + \delta)^{n+k_1}}{r_1^{3+2\gamma}} = \frac{1}{r_1^3} \left[\left(\frac{\rho + \delta}{r_1} \right)^{2\gamma} (\rho + \delta)^{n+1k-2\gamma} \right]; \; n + k_1 - 2\gamma \geqq 0 .$$

Für die Abschätzung von $\frac{\rho+\delta}{r_1}$ folgt zunächst aus $r_1^2=(1-r)^2+u$

$$\frac{\delta^2}{r_1^2} \leqq \frac{\delta^2}{u} = \frac{\delta^2}{\frac{16r\delta^2}{(4+\rho')^2(4+\rho^2)}}$$

$$= \frac{(4+\rho'^2)(4+\rho^2)}{16r} \to \frac{4+\rho'^2}{4} \text{ für } p \to P_0. \tag{40}$$

Ferner ist

$$\frac{\rho^2}{r_1^2} = \frac{\rho^2}{(1-r)^2+u} \leqq \frac{\rho^2}{(1-r)^2}, \text{ weil } u = \frac{16r\delta^2}{(4+\rho'^2)(4+\rho^2)} \geqq 0.$$

Um den Bruch $\frac{\rho}{1-r}$ abzuschätzen, betrachten wir das Dreieck CpP_0.

Sei $\sphericalangle CP_0p=\theta$. Nach Voraussetzung ist $\theta \leqq \Theta < \frac{\pi}{2}$. Ferner ist $\sphericalangle P_0Cp=\vartheta$ und $\overline{Cp}=r$. Nach dem Sinussatz ist

$$r = \frac{\sin\theta}{\sin(\pi-\theta-\vartheta)} = \frac{\sin\theta}{\sin(\theta+\vartheta)} \leqq \frac{\sin\Theta}{\sin(\Theta+\vartheta)},$$

wenn ϑ genügend klein ist. Da ferner $\rho = 2\operatorname{tg}\vartheta_{/2} = \frac{2\sin\vartheta}{1+\cos\vartheta}$, so ist

$$\frac{\rho}{1-r} \leqq \frac{2\sin\vartheta}{1+\cos\vartheta} \cdot \frac{(\sin\Theta+\vartheta)}{\sin(\Theta+\vartheta)-\sin\Theta}.$$

Dieser Ausdruck bleibt beschränkt für $\vartheta>0$ und *strebt für* $\vartheta \to 0$ *gegen* $\operatorname{tg}\Theta$, wie eine einmalige Differentiierung von Zähler und Nenner nach ϑ zeigt.

Aus diesen Abschätzungen folgt, dass $\frac{\rho}{r_1}$ und $\frac{\delta}{r_1}$ beschränkt bleiben, dass es also eine positive, endliche Zahl M'_{rk} gibt, derart dass

$$M'_{rk} > \left| C_{rk}(r,p,P') \cdot \frac{(\rho+\delta)^{2r}}{r_1^{2r}} \cdot (\rho+\delta)^{n+k_1-2r} \right|,$$

und ebenso eine positive, endliche Zahl M_n, derart, dass

$$M'_n = \sum_{r,k} M_{rk},$$

und, dass folglich unter Beachtung der Formel (37) gilt

$$\left| \frac{1}{4\pi} \int_{\omega(P_0,P') \leqq \varepsilon} \rho'^n \Omega d^n F(r,\omega) d\sigma_{P'} \right|$$

$$< M'_n \frac{1-r^2}{4\pi} \int_{\omega(P_0,P') \leqq \varepsilon} \frac{|\Omega|}{r_1^3} d\sigma_{P'} < M'_n \overline{\Omega}(P_0,\varepsilon).$$

Der letzte Ausdruck strebt für $\varepsilon \to 0$ nach Voraussetzung gegen Null. Damit ist Theorem XI vollständig bewiesen.

Die Beschränkung in bezug auf die Art der Annäherung $p \to P_0$, die wir bis jetzt festhalten mussten, kann fallen gelassen werden, wenn in bezug auf $V(P)$ folgende Voraussetzung gemacht wird:

§ 3. $V(P)$ besitzt in P_0 und Umgebung ein Differential n^{ter} Ordnung, welches in P_0 stetig ist.

Theorem XII. Wenn die Randfunktion $V(P)$ in einem Punkte P_0 und in allen Punkten P der Umgebung von P_0 ein Differential n^{ter} Ordnung besitzt, und dieses Differential im Punkte P_0 stetig ist, so strebt das n^{te} Differential des Poissonschen Integrals $v(p)$ bei jeder beliebigen Annäherung $p \to P_0$ in K gegen den Wert des n^{ten} Differentials von $V(P)$ im Punkte P_0.

Es werden also zwei Dinge vorausgesetzt: 1. dass das n^{te} Differential von $V(P)$ *in P_0 und in einer Umgebung ε von P_0 existiere, d.h.* dass in der Umgebung eines jeden Punktes P der Kalotte $\omega(P_0, P) \leqq \varepsilon$, sich $V(P')$ darstellen lasse durch die (in § 2 näher erklärte) Formel

$$V(P') = V_n(P, P') + \delta^n \, \Omega(x, y\,; x', y')\,; \qquad \omega(P_0, P) \leqq \varepsilon, \tag{41}$$

wo $$V_n(P, P') = v(x, y) + \sum_{r=1}^{n} \frac{1}{r\,!} \left[(x'-x) \frac{\partial}{\partial x} + (y'-y) \frac{\partial}{\partial y} \right]^{(r)} v(x, y),$$

$$\delta^2 = (x'-x)^2 + (y'-y)^2$$

und $$\overline{\Omega}(P, \eta) = \text{obere Schranke} \mid \Omega(x, y\,; x', y') \mid \to 0 \text{ für } \eta \to 0,$$

$$0 \leqq \omega(P, P') \leqq \eta.$$

Nach der letzten Beziehung, in welcher Ω als Funktion von $(x'y')$ aufgefasst wird, gibt es für jeden Punkte P der Umgebung ε von P_0 zu jedem $\eta_1 > 0$ ein $\eta_P > 0$, derart, dass

$$\overline{\Omega}(P, \eta) < \eta\,, \text{ wenn } \eta \leqq \eta_P.$$

Es wird 2. vorausgesetzt, dass das n^{te} Differential von $V(P)$ *im Punkte P_0 stetig* sei. Wir sagen definitionsweise, dass dies dann der Fall sei, wenn alle Ableitungen $\dfrac{\partial^{h+k} v(x, y)}{\partial x^h \partial y^k} (h + k \leqq n)$ im Punkte P_0 stetig sind, und wenn überdies $\overline{\Omega}(P, \eta)$ in der Umgebung von P_0 *gleichmässig* gegen Null strebt, d.h. zu jedem $\eta_1 > 0$ lässt sich ein von P unabhängiges $\eta_2 > 0$ derart bestimmen, dass

$$\overline{\Omega}(P, \eta) < \eta_1 \tag{42}$$

für alle P der Kalotte $\omega(P_0, P) \leqq \eta_2$ und für alle η, die der Ungleichung

$\eta \leqq \eta_2$ genügen. Wir wählen $\varepsilon = \eta_2$. Dann gilt *Ungleichung* (42) *für alle* P *der Kalotte* $\omega(P_0, P) \leqq \varepsilon$ *und alle* P' *der Kalotte* $\omega(P, P') \leqq \varepsilon$.

Um den *Beweis* des Theorems zu führen, betrachten wir eine Kalotte $\Gamma \equiv \omega(P, P') \leqq \varepsilon$, deren Mittelpunkt P auf dem gleichen Radius mit p liegt und folglich mit p gegen P_0 rückt. Für jede Lage von p gilt

$$d^n v(p) = \frac{1}{4\pi} \int_\Gamma V(P') d^n F(r, \omega)\, d\sigma_{P'} + \frac{1}{4\pi} \int_{S-\Gamma} V(P') d^n F(r, \omega) d\sigma_{P'}.$$

Das zweite Integral der rechten Seite dieser Gleichung strebt bei Annäherung an einen beliebigen Punkt P von S gleichmässig gegen Null. (Theorem X. Formel 25a.). Es ist also

$$\lim_{p \to P_0} \frac{1}{4\pi} \int_{S-\Gamma} V(P') d^n F(r, \omega)\, d\sigma_{P'} = 0.$$

Das erste Integral der vorletzten Gleichung lässt sich, wenn P nahe genug an P_0 gerückt ist nach Voraussetzung (41) auf folgende Form bringen:

$$\frac{1}{4\pi} \int_\Gamma V(P') d^n F(r, \omega)\, d\sigma_{P'} = \frac{1}{4\pi} \int_\Gamma V_n(P, P')\, d^n F(r, \omega)\, d\sigma_{P'}$$
$$+ \frac{1}{4\pi} \int_\Gamma \delta^n \Omega d^n F(r, \omega)\, d\sigma_{P'};\ \omega(P_0, P) \leqq \varepsilon. \quad (43)$$

Beachten wir beim ersten Integral der rechten Seite, dass V_n zwar analytisch ist, aber nicht nur von P', sondern auch von P abhängt, und dass anderseits die Kalotte Γ ihre Lage mit P ändert, sodass Theorem IX nicht ohne weiteres Anwendung findet. Lassen wir aber P nahe genug an P_0 heranrücken, z. B. so, dass $\omega(P_0, P) \leqq \frac{\varepsilon}{4}$, dann ist und bleibt die feste Kalotte Γ_0 mit Mittelpunkt P_0 und Radius $\frac{\varepsilon}{2}$ ganz im Innern der veränderlichen Kalotte Γ. Anderseits haben alle Punkte P' der veränderlichen Kalotte Γ, welche ausserhalb der festen Kalotte Γ_0 liegen, den Abstand $\omega(P, P') \geqq \frac{\varepsilon}{4}$. Wir nennen die Menge dieser Punkte $\mathfrak{M}$. Wir erhalten dann für das erste Integral auf der rechten Seite der Gleichung (44)

$$\frac{1}{4\pi} \int_\Gamma V_n(P, P')\, d^n F(r, \omega)\, d\sigma_{P'} = \frac{1}{4\pi} \int_{\Gamma_0} V_n(P, P') d^n F(r, \omega)\, d\sigma_{P'}$$

$$+\frac{1}{4\pi}\int_{\mathfrak{M}} V_n(P,P')\, d^n F(r,\omega)\, d\sigma_{P'};\ \omega(P_0;P) \leqq \frac{\varepsilon}{4}.$$

Auf das letzte Integral findet gleichmässig in P die Formel (25c) Anwendung, sodass

$$\lim_{p\to P_0} \frac{1}{4\pi}\int_{\mathfrak{M}} V_n(P,P')\, d^n F(r,\omega)\, d\sigma_{P'} = 0.$$

Um den Grenzwert des ersten Integrals für $p\to P_0$,

$$\lim_{p\to P_0} \frac{1}{4\pi}\int_{\Gamma_0} V_n(P,P')\, d^n F(r,\omega)\, d\sigma_{P'}, \tag{44}$$

zu bestimmen, zerlegen wir $V_n(P,P')$ in seine Bestandteile. Aus

$$V_n(P,P') = v(x,y) + \sum_{\nu=1}^{n} \frac{1}{\nu!}\left[(x'-x)\frac{\partial}{\partial x} + (y'-y)\frac{\partial}{\partial y}\right]^{(\nu)} v(x,y)$$

entstehen zunächst, das Glied $v(x,y)$ abgerechnet, Ausdrücke von der Form

$$\frac{1}{\nu!}\binom{\nu}{\mu}(x'-x)^{\nu-\mu}(y'-y)^{\mu}\frac{\partial^\nu v(x,y)}{\partial x^{\nu-\mu}\,\partial y^{\mu}};\ \mu=0,1,2\ldots.\nu;\ \nu=1,2\ldots.n,$$

und durch Ausführung der Potenzen $(x'-x)^{\nu-\mu}$ und $(y'-y)^{\mu}$

$$\frac{1}{\nu!}\binom{\nu}{\mu}\binom{\nu-\mu}{\sigma}\binom{\mu}{t}(-1)^{\sigma+\tau}x'^{\nu-\mu-\sigma}x^{\sigma}y'^{\mu-\tau}y^{\tau}\,\frac{\partial^\nu v(x,y)}{\partial x^{\nu-\mu}\,\partial y^{\mu}};$$

$$\sigma=0,1,2\ldots.(\nu-\mu);\ \tau=0,1,2,\ldots.\mu.$$

Die Grössen x^σ, y^τ und $\frac{\partial^\nu v(x,y)}{\partial x^{\nu-\mu}\partial y^{\mu}}$ sind von den Integrationsvariablen unabhängig und können als Faktoren vor das Integral gesetzt werden. Die Ableitungen $\frac{\partial^\nu v(x,y)}{\partial x^{\nu-\mu}\,\partial y^{\mu}}$ streben für $p\to P_0$ nach Voraussetzung stetig gegen $\frac{\partial^\nu v(0,0)}{\partial x^{\nu-\mu}\,\partial y^{\mu}}$; weil ferner x und y für $p\to P_0$ gegen Null streben, so verschwinden alle Teilintegrale, in welchen $\sigma+\tau>0$ ist. Es bleiben also für (44) Ausdrücke von der Form

$$\frac{1}{\nu!}\binom{\nu}{\mu}\frac{\partial^\nu v(0,0)}{\partial x^{\nu-\mu}\partial y^{\mu}}\cdot\lim_{p\to P_0}\frac{1}{4\pi}\int_{\Gamma_0} x'^{\nu-\mu}y'^{\mu}\, d^n F(r,\omega)\, d\sigma_{P'};$$

$$\mu=0,1,2\ldots.\nu;\ \nu=1,2,\ldots.n.$$

Der Grenzwert eines solchen Ausdrückes ist nach Theorem IX

$$\frac{1}{\nu!}\binom{\nu}{\mu}\frac{\partial^\nu v(0,0)}{\partial x^{\nu-\mu}\partial y^{\mu}}\cdot d^n(x'^{\nu-\mu}y'^{\mu})P_0.$$

Nun ist aber

$$d^n(\alpha'^{\nu-\mu}y'^{\mu})=\begin{cases}0, \text{ wenn } \nu<n;\\(n-\mu)!\,\mu!\,dx^{n-\mu}dy^{\mu}, \text{ wenn } \nu=n.\end{cases}$$

Somit erhalten wir für (44) die Summe

$$\sum_{\mu=0}^{n}\frac{1}{n!}(n-\mu)!\,\mu!\,\binom{n}{\mu}\binom{n}{\mu}dx^{n-\mu}dy^{\mu}\frac{\partial^n v(0,0)}{\partial x^{n-\mu}\partial y^{\mu}}=d^n v(0,0).$$

Das noch vernachlässigte Glied $v(x,y)$ spielt keine Rolle, denn das darauf bezügliche Integral ist Null, wie sich ergibt, wenn man $\nu=0$ setzt.—Es ist also

$$\lim_{p\to P_0}\frac{1}{4\pi}\int_{\Gamma_0}V_n(P,P')\,d^nF(r,\omega)\,d\sigma_{P'}=d^nv(0,0)=d^nV(P_0). \tag{45}$$

Es bleibt nur noch das zweite Integral der Gleichung (43) abzuschätzen oder zu beweisen, dass

$$\lim_{p\to P_0}\frac{1}{4\pi}\int_{\Gamma}\delta_n\Omega\,d^nF(r,\omega)d\sigma_{P'}=0;\ \ \omega(P_0,P)<\varepsilon. \tag{46}$$

Nach (38) gilt für einen beliebigen Punkt p

$$\delta^n\,.\,d^nF(r,\omega)=(1-r^2)\sum_{\nu,k}C_{\nu k}(r,p,P')\,.\,\frac{\delta^{n+k_1}}{r_1^{\,3+2\nu}},$$

wo $C_{\nu k}(r,p,P')$ eine beschränkte Funktion ist und die Ungleichung gilt $k_1+n-2\nu\geqq 0$. Aus Ungleichung (40) folgt ferner

$$\left(\frac{\delta}{r_1}\right)^{2\nu}\leqq\left[\frac{(4+\rho'^2)(4+\rho^2)}{16r}\right]^{2\nu}.$$

Es gibt auch hier wieder eine Zahl M'', derart dass

$$M''_{\nu k}>\left|C_{\nu k}(r,p,P').\frac{\delta^{2\nu}}{r_1^{\,2\nu}}\,\delta^{n+\nu,-2\nu}\right|$$

und ebenso eine Zahl $M_n''=\sum\limits_{\nu,k}M''_{\nu k}$, sodass (unter Berücksichtigung von (37) gilt

$$\left|\frac{1}{4\pi}\int_{\Gamma}\delta^n\Omega\,d^nF(r,\omega)\,d\sigma_{P'}\right|<M''_n\frac{1-r^2}{4\pi}\int_{\Gamma}\frac{|\Omega|}{r_1^{\,3}}d\sigma_{P'}\leqq M''_n\overline{\Omega}(P';\varepsilon).$$

Nun ist aber, weil $\omega(P_0,P)<\varepsilon$ und $\omega(P,P')\leqq\varepsilon$, ferner $\varepsilon=\eta_2$, Ungleichung (42) anwendbar, wonach $\overline{\Omega}(P;\varepsilon)<\eta_1$, also beliebig klein ist. Der letzte Ausdruck strebt also für $P\to P_0$ gegen Null. Für das Differential $d^nv(p)$ bleibt also bei einer beliebigen Annäherung $p\to P_0$ in K nur der in Formel (45) dargestellte Ausdruck $d^nV(P_0)$ übrig, was dem zu beweisenden Theorem entspricht.

Aus den dargelegten Beweisen ergibt sich folgendes *Korollarium. Wenn das Differential $d^n V(P)$ in allen Punkten eines Bereiches stetig ist, so strebt das Differential $d^n v(p)$ bei beliebiger Annäherung $p \to P_0$ in K an einen Punkt P_0 dieses Bereiches gleichmässig gegen $d^n V(P_0)$.*

§ **4.** $V(P)$ besitzt in P_0 eine verallgemeinerte Ableitung.

Das Verhalten der Ableitungen von $v(p)$ bei Annäherung $p \to P_0$ in K kann, wie in der Einleitung bemerkt wurde, auch durch direkte und vollständige Berechnung dieser Ableitungen untersucht werden. Dieser Weg führt zwar zu sehr allgemeinen Resultaten, in welchen die verallgemeinerten Ableitungen von $V(P)$ vorkommen, aber er ist praktisch nur für die Ableitungen der ersten Ordnungen gangbar, weil er sich rasch in sehr langen und komplizierten Rechnungen verliert. Wir werden in diesem Paragraphen diesen Weg und die entsprechenden Resultate nur skizzieren. Eine ausführliche Darstellung kann schon deswegen erspart werden, weil sich eine solche fast wörtlich mit einer analogen, von Herrn Plancherel über die Laplacesche Reihe durchgeführten Untersuchung(1) decken würde. Wir beschränken die Untersuchung auf *radiale Annäherung* und auf die partiellen Ableitungen der *ersten und zweiten Ordnung* in bezug auf die rechtwinkligen Koordinaten (x, y), welche den im dritten Kapitel erklärten Sinn haben. Das Achsensystem wählen wir so, dass der Punkt P_0 in den Pol zu liegen kommt. Um unnötige Wiederholungen zu vermeiden, werden wir, gestützt auf Theorem X (Formel 25a), das Integral über die Kalotte $\omega(P_0, P') > \varepsilon$ vernachlässigen und nur folgenden Ausdruck untersuchen:

$$\lim_{r \to 1} \frac{1}{4\pi} \int_{\omega(P_0, P') \leq \epsilon} V(P') \, D^n_{i,k} F(r, \omega)_0 \, d\sigma_{P'}\,; \quad n = 1, 2,$$

wo $D^n_{i,k} F(r, \omega)_0$ die partielle Ableitung von $F(r, \omega)$ für $x = y = 0$ von der Ordnung i in bezug auf x und von der Ordnung k in bezug auf y ist $(i + k = n)$. Wenn dieser Grenzwert existiert, so ist er gleich $\lim\limits_{p_0 \to P_0} D^n_{i,k} v(p_0)$.

Eine einfache Rechnung ergibt aus (23) und (24):

$$\left.\begin{aligned} &\left(\frac{\partial \omega}{\partial x}\right)_{x=y=0} = -\cos \varphi'\,; \qquad \left(\frac{\partial \omega}{\partial y}\right)_{x=y=0} = -\sin \varphi'\,; \\ &\left(\frac{\partial^2 \omega}{\partial x^2}\right)_{x=y=0} = -\operatorname{cotg} \vartheta' \sin^2 \varphi'\,; \left(\frac{\partial^2 \omega}{\partial y^2}\right)_{x=y=0} = \operatorname{cotg} \vartheta' \cos^2 \varphi'\,; \end{aligned}\right\} \quad (47)$$

(1) M. Plancherel, Sur la sommation des séries de Laplace et de Legendre, Rendiconti del Circ. Mat. di Palermo 33 (1912), p. 1–16.—Dieser Arbeit sind auch die Definitionen der verallgemeinerten Ableitungen entnommen.

$$\left(\frac{\delta^2\omega}{\delta x \delta y}\right)_{x=y=0} = -\operatorname{cotg} \vartheta' \cos\varphi' \sin\varphi'.$$

Ferner lassen sich durch partielle Integration unter Berücksichtigung von (2) und (5) folgende *Hilfsformeln* ableiten:

$$\lim_{r\to 1} \frac{1}{2}\int_0^\varepsilon \frac{dF(r,\vartheta')}{d\vartheta'} \sin^2\vartheta'\, d\vartheta' = -2, \tag{48}$$

$$\lim_{r\to 1} \frac{1}{2}\int_0^\varepsilon \frac{d^2F(r,\vartheta')}{d\vartheta'^2} \sin^3\vartheta'\, d\vartheta' = 2.3 = 6. \tag{49}$$

Durch Einsetzen des in Formel (27) gegebenen Wertes erhalten wir, vorausgesetzt, dass der Grenzwert der rechten Seite der folgenden Gleichung existiert, für die *Ableitung erster Ordnung in bezug auf x*:

$$\lim_{r\to 1} D^1_{1,0}\, v(p_0) = -\frac{1}{4\pi}\int_{\omega(P_0,P')\leqq\varepsilon} V(P')\, \frac{dF(r,\vartheta')}{d\vartheta'} \sin\vartheta'\, d\vartheta' \cos\varphi'\, d\varphi'$$

$$= -\frac{1}{2}\int_0^\varepsilon \frac{dF(r,t)}{dt} \sin^2 t\, dt\, \frac{1}{\pi \sin t}\int_{\vartheta'=t} V(P') \cos\varphi'\, d\varphi'. \tag{50}$$

Unter der Voraussetzung, dass das Linienintegral

$$V_{1,0}(P_0;t) = \frac{1}{\pi\sin t}\int_{\vartheta'=t} V(P')\cos\varphi'\, d\varphi'$$

für $t\to +0$ einen Grenzwert

$$V_{1,0}(P_0;+0) = \lim_{t\to+0} \frac{1}{\pi\sin t}\int_{\vartheta'=t} V(P')\cos\varphi'\, d\varphi'$$

besitzt, nennen wir $V_{1,0}(P_0;+0)$ *die verallgemeinerte Ableitung von $V(P)$ in bezug auf x im Punkte P_0.*

Wenn diese Voraussetzung erfüllt ist, so lässt sich unter Berücksichtigung der Formel (48) und unter Anwendung der im ersten Kapitel mehrfach durchgeführten Schlussweise aus (50) folgendes Resultat ableiten:

$$\lim_{r\to 1} D^1_{1,0} v(p_0) = V_{1,0}(P_0;+0) = \lim_{t\to+0} \frac{1}{\pi\sin t}\int_{\vartheta'=t} V(P')\cos\varphi'\, d\varphi',$$

d.h. wenn die verallgemeinerte Ableitung von $V(P)$ in bezug auf x im Punkte P_0 existiert, so strebt die Ableitung von $v(p)$ in bezug auf x bei radialer Annäherung $p_0\to P_0$ nach dem Wert der genannten verallgemeinerten Ableitung von $V(P)$. Weil wir jede beliebige Richtung als x-Richtung annehmen können, so gilt dieser Satz ganz allgemein für jede Richtung. Bildet eine Richtung die wir die φ-Richtung nennen, mit der angenommen x-Richtung den Winkel φ, so lässt sich die verallgemeinerte

Ableitung in bezug auf diese φ—Richtung durch einfache Überlegungen auf folgende Form bringen:

$$V_\varphi(P_0\,;\,+0) = \lim_{t\to+0} \frac{1}{\pi \sin t} \int_{\vartheta'=t} V(P') \cos(\varphi'-\varphi)\, d\varphi'.$$

Völlig analoge Überlegungen führen unter Anwendung der Formel (49) zu entsprechenden Resultaten in bezug auf die *Ableitungen zweiter Ordnung.* Sind $\varphi_1 = \varphi$ und $\varphi_2 = \varphi + \frac{\pi}{2}$ zwei beliebige aufeinander senkrechte Richtungen im Punkte P_0, $D^2_{\varphi\varphi} v(p)$ die zweite Ableitung von $v(p)$ in bezug auf die Richtung φ und $D^2_{\varphi_1\varphi_2} v(p)$ die Ableitung zweiter Ordnung in bezug auf die beiden aufeinander senkrechten Richtungen φ_1 und φ_2, so gilt, wenn in den folgenden Gleichungen der Grenzwert der rechten Seite als existierend vorausgesetzt wird:

$$\lim_{r\to 1} D^2_{\varphi\varphi} v(p_0) = V_{\varphi\varphi}(P_0; +0)$$

$$= \lim_{t\to+0} \frac{1}{\pi \sin^2 t} \int_{\vartheta'=t} [V(P') - V(+0)][3\cos^2(\varphi'-\varphi) - \sin^2(\varphi'-\varphi)]\, d\varphi' \text{ (1)}.$$

$$\lim_{r\to 1} D^2_{\varphi_1\varphi_2} v(p_0) = V_{\varphi_1\varphi_2}(P_0; +0)$$

$$= \lim_{t\to+0} \frac{1}{\pi \sin^2 t} \int_{\vartheta'=t} [V(P') - V(+0)]\, 2\sin(\varphi - \varphi)\, d\varphi'.$$

Wir nennen $V_{\varphi\varphi}(P_0; +0)$ und $V_{\varphi_1\varphi_2}(P_0; +0)$ die verallgemeinerten Ableitungen zweiter Ordnung von $V(P)$ in bezug auf die Richtungen φ bezw. φ_1 und φ_2.

Bei der Definition der verallgemeinerten Ableitungen ist immer vorausgesetzt, dass P_0 im Pol des Achsensystems liegt. Offenbar lassen sich die Definitionen auf beliebige Punkte ausdehnen.—Die Resultate dieses Paragraphen lassen sich in folgendes Theorem zusammenfassen:

Theorem XIII. Wenn die Randfunktion $V(P)$ im Punkte P_0 verallgemeinerte Ableitungen erster oder zweiter Ordnung im oben definierten Sinne besitzt, so streben die partiellen Ableitungen von $v(p)$ bei radialer Annäherung an P_0 gegen den Wert der entsprechenden verallgemeinerten Ableitungen in diesem Punkte.

§ 5. Anwendung der gefundenen Resultate auf Reihe von Laplace.

Wie in der Einleitung bemerkt wurde, bildet das Poissonsche

(1) $V(+0)$ bedeutet hier den Kreismittelwert von $V(P)$ im Punkte P_0.

Integral ein Mittel, um die Reihe von Laplace zu summieren. Entwickelt man nämlich $F(r, \omega)$ nach Potenzen von r, so erhält man für $v(p)$ die Reihe (¹),

$$v(p) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} r^n \int_S V(P')P_n(\cos\omega)d\sigma_{P'} \equiv \sum_{n=0}^{\infty} r^n Y_n,$$

wo $P_n(\cos\omega)$ das Legendresche Polynom vom Grade n in $\cos\omega$ bedeutet. Aus der Ungleichung $|P_n(\cos\omega)| \leqq 1$ folgt:

$$|Y_n| = \left| \frac{2n+1}{4\pi} \int_S V(P')P_n(\cos\omega)d\sigma_{P'} \right|$$
$$\leqq \frac{2n+1}{4\pi} \int_S |V(P')| d\sigma_{P'} \leqq \frac{(2n+1)M}{4\pi}.$$

Die Potenzreihe $\Sigma r^n Y_n$ ist also für $r < 1$ gleichmässig in ϑ und φ konvergent, wenn nur $V(P)$ integrierbar ist.

Unter der *Laplaceschen Reihe* von $V(P)$ versteht man die Reihe $\sum\limits_{n=0}^{\infty} Y_n$. Diese Reihe braucht nicht zu konvergieren.

Unter *Poissonscher Summation* der Laplaceschen Reihe versteht man den Grenzwert

$$\lim_{r \to 1} \sum_{n=0}^{\infty} r^n Y_n.$$

Dieser Limes kann bestehen, ohne dass die Laplacesche Reihe konvergiert. Wenn aber in einem Punkte (ϑ, φ) die Laplacesche Reihe konvergiert, so ist nach einem Abelschen Satze folgende Gleichung erfüllt:

$$\sum_{n=0}^{\infty} Y_n(\vartheta, \varphi) = \lim_{r \to 1} \sum_{n=0}^{\infty} r^n Y_n(\vartheta, \varphi).$$

Wenden wir nun die gefundenen Resultate auf die Poissonsche Summation der Laplaceschen Reihe an. Nach Theorem V gilt folgendes: *Wenn die Funktion $V(P)$ im Punkte P_0 einen Kalottenmittelwert $V_2(P_0)$ besitzt, so ist*

$$\lim_{r \to 1} \sum_{n=0}^{\infty} r^n Y_n(P_0) = V_2(P_0).$$

Auf gleiche Weise lassen sich Theorem I bis IV und Theorem VI übertragen.

Ferner gilt in bezug auf die *Differentiale* nach Theorem XI und XII

(¹) Vgl. E. Picard, Traité d'analyse, Paris (1891) I, p, 200.

und dem zugehörigen Korollarium folgender Satz: *Ist* $\sum\limits_{n=0}^{\infty} Y_n$ *die formell gebildete Laplacesche Reihe der Funktion* $V(P)$, *und besitzt* $V(P)$ *im Punkte* P_0 *ein Differential* n^{ter} *Ordnung, so ist*

$$\lim_{r\to 1}\sum_{\nu=0}^{\infty} r^{\nu}\,\frac{\partial^n Y_\nu(P_0)}{\partial\vartheta^k\,\partial\varphi^{n-k}}=\frac{\partial^n V(P_0)}{\partial\vartheta^k\,\partial\varphi^{n-\kappa}}.$$

Diese Formel gilt gleichmässig für alle Punkte eines Bereiches, auf welchem das n^{te} *Differential überall stetig ist.*

An Extension of a Theorem of Salmon,

by

Yûzaburô Sawayama, Tôkyô.

Theorem: Take any two points A, B and their polars with respect to a fixed conic S. Let the perpendiculars, drawn from each of A and B to its own polar, meet an axis of the conic in the points M and N respectively, and let the perpendiculars drawn from each of A and B to the polar of the other point be AP and BQ. Then

$$AM \cdot BQ = BN \cdot AP,$$

each segment of line having the proper sign according to the rule of analytical geometry.

Proof: Taking the axis MN and the one of its conjugate chords as the axis of coordinates, we get the equation to the conic of the form

$$ax^2 + y^2 + 2gx + c = 0.$$

Let the coordinates of the points A and B be (x', y') and (x'', y'') respectively. Then the polar of the point A is

$$ax'x + y'y + g(x' + x) + c = 0. \qquad (1)$$

If D is the intersection of the line AM and the polar (1), we have

$$\frac{BQ}{AD} = \frac{ax'x'' + y'y'' + g(x' + x'') + c}{ax'^2 + y'^2 + 2gx' + c}, \qquad (2)$$

since BQ is parallel to AD.

If the perpendicular AE drawn from A to the axis MN meets the polar (1) at the point F, then the coordinates of $F(x', y' + AF)$ will satisfy the equation (1), namely

$$ax'^2 + y'(y' + AF) + 2gx' + c = 0,$$

so we have

$$-y' \,.\, AF = ax'^2 + y'^2 + 2gx' + c. \qquad (3)$$

Since the points D, M, E, F are concyclic, we see that

$$-y' \cdot AF = AE \,.\, AF = AM \cdot AD,$$

and hence from (3)

$$AM \cdot AD = ax'^2 + y'^2 + 2gx' + c.$$

From this and (2), we finally get

$$AM \cdot BQ = AM \cdot AD \cdot \frac{BQ}{AD}$$

$$= ax'x'' + y'y'' + g(x' + x'') + c.$$

Since the last expression is symmetric with respect to the points A and B, we must have

$$AM \cdot BQ = BN \cdot AP.$$

N.B. The above theorem still holds when we take any diameter of the conic instead of the axis, provided that AM and BQ are antiparallel to any conjugate chord of the diameter with respect to the angle between the diameter and the polar of A, and BN and AP similarly modified.

Specially if the conic S is the circle, the two points M and N coincide with the center, and we get the theorem of Salmon.

Über eine arithmetische Eigenschaft gewisser Reihenentwicklungen,

(Auszug aus einem an Herrn M. Fujiwara gerichteten Briefe),

von

GEORG PÓLYA in Zürich (Schweiz).

....Ihr Satz über die Periodizität der Entwicklungskoeffizienten mod. m(1) hat mich seit längerer Zeit ausserordentlich interessiert, denn er schien auf einen eigenartigen und von anderer Seite her unbekannten Zusammenhang zwischen arithmetischen Eigenschaften und analytischem Charakter hinzuweisen. Ich bemühte mich die Frage zu entscheiden, ob Ihr Satz nicht von beschränkenden Voraussetzungen befreit und in folgender Form ausgesprochen werden kann: *Sind $a_0, a_1, a_2, \ldots . a_n, \ldots .$ ganze Zahlen, und genügt die Reihe*

$$a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \ldots . + \frac{a_n}{n!}x^n + \ldots . \tag{1}$$

einer algebraischen Differentialgleichung, so ist die Zahlenfolge $a_0, a_1, a_2, \ldots . a_n, \ldots .$ von einer gewissen Stelle an periodisch nach jedem Modul m, der durch gewisse, in endlicher Anzahl vorhandene Ausnahme-Primzahlen nicht teilbar ist. Leider ist aber der Satz in dieser Allgemeinheit nicht richtig, sondern es müssen Beschränkungen dazu kommen, wie ich es sofort mit einem Beispiel belegen werde, und zwar lassen sich die von Ihnen a. a. O. S. 58–59 angegebenen Beschränkungen nicht wesentlich erweitern.

Soll die Reihe

$$y = A_0 + A_1 x + A_2 x^2 + \ldots .$$

der Differentialgleichung

$$xy'' + (1-4x)y' - 2y = 0 \tag{2}$$

genügen, so muss für $n = 1, 2, 3, \ldots .$

$$n(n-1)A_n + nA_n - 4(n-1)A_{n-1} - 2A_{n-1} = 0,$$

$$A_n = \frac{4n-2}{n^2}A_{n-1} = \frac{2n(2n-1)}{n^3}A_{n-1}$$

---

(1) M. Fujiwara, Über die Periodizität der Entwicklungskoeffizienten einer analytischen Funktion nach dem Modul m, dieses Journal Bd. 2, S. 57–73 (1912).

sein, also, $A_0=1$ gesetzt,

$$A_n=\frac{2n\,!}{n\,!^3}=\frac{1}{n\,!}\binom{2n}{n}.$$

Der Binomialkoeffizient $\binom{2n}{n}$ ist eine ganze Zahl, also die Reihe

$$(3) \qquad 1+\binom{2}{1}\frac{x}{1!}+\binom{4}{2}\frac{x^2}{2!}+\ldots+\binom{2n}{n}\frac{x^n}{n!}+\ldots$$

ist eine Reihe von dem Typus (1), eine „ganzzahlige Reihe" im Sinne von Hurwitz. Die Reihe (3) genügt der Differentialgleichung (2), *aber die Folge der Koeffizienten*

$$(4) \qquad 1,\ \binom{2}{1},\ \binom{4}{2},\ \binom{6}{3},\ldots\binom{2n}{n},\ldots$$

ist nach keinem ungeraden Primzahlmodul periodisch (weder rein-, noch gemischt-periodisch), wie ich es sofort zeigen will.

Es sei p eine Primzahl, $p\geqq 3$.

Es sei gegeben eine positive ganze Zahl k. (Der springende Punkt wird sein, dass k beliebig gross gewählt werden kann.) Die höchste Potenz von p, die in $(2k-1)!$ aufgeht, sei bezeichnet mit p^{r-1}. Dieser Definition gemäss ist $r\geqq 1$. Es sei

$$(5) \qquad p^r=m$$

gesetzt. Es ist für $\qquad q\geqq 1$

$$\binom{2m}{m}=\frac{2}{1}\,\frac{2m-1}{m}\,\frac{2m-2}{m-1}\,\frac{2m-3}{m-1}\cdots\times$$

$$\frac{2m-2q+2}{m-q+1}\,\frac{2m-2q+1}{m-q+1}\binom{2(m-q)}{m-q}$$

(der erste Bruch rechts ist mit m gekürzt). Daraus folgt, gemäss (5)

$$-2.\ (2q-1)!\binom{2(m-q)}{m-q}\equiv 0 \ (\text{mod. } p^r).$$

Falls $q\leqq k$, so kann man des weiteren behaupten, gemäss der Definition der Zahl r, dass

$$\binom{2(m-q)}{m-q}\equiv 0 \ (\text{mod. } p).$$

Es ist $\binom{2m}{m}=\dfrac{2m!}{m!\,m!}$ durch p^α teilbar und durch $p^{\alpha+1}$ nicht mehr teilbar, wenn

$$\alpha=\left[\frac{2m}{p}\right]+\left[\frac{2m}{p^2}\right]+\left[\frac{2m}{p^3}\right]+\ldots-2\left(\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\left[\frac{m}{p^3}\right]+\ldots\right)$$

gesetzt wird, nach einer bekannten Regel. Gemäss (5) ist aber die so definierte Zahl $\alpha=0$. Anders gesagt, es ist

$$\binom{2m}{m} \not\equiv 0 \pmod{p}.$$

Andererseits haben wir eben festgestellt, dass

$$\binom{2(m-k)}{m-k} \equiv \binom{2(m-k+1)}{m-k+1} \equiv \dots \equiv \binom{2(m-1)}{m-1} \equiv 0 \pmod{p}.$$

Wird die Zahlenfolge (4) mod. p auf die kleinsten nichtnegativen Reste reduziert, so befinden sich darin *beliebig lange* Sequenzen von aneinanderfolgenden Gliedern, die $=0$ sind, gefolgt von einem Glied, das $\neq 0$ ist: also ist weder die Folge (4) mod. p periodisch, noch irgend eine Folge, die aus (4) durch Wegwerfen von endlich vielen Gliedern entsteht, w. z. b. w.

Ihrem Satz a.a. O. widerspricht mein Beispiel nicht. Wird nämlich die linke Seite von (2) mit $F(x, y, y', y'')$ bezeichnet, so ist

$$\frac{\partial F}{\partial y''} = x, \qquad \frac{\partial F}{\partial y'} = 1 - 4x,$$

also ist keine der beiden Bedingungen erfüllt, unter denen Sie die Periodizität der Koeffizienten $a_0, a_1, a_2, \dots$ in der Reihe (1) gezeigt haben.

Ich habe übrigens das Beispiel von einer anderen Fragestellung ausgehend gefunden. Sind $a_0, a_1, a_2, \dots$ ganze Zahlen, und stellt die Reihe

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

eine rationale Funktion dar, so ist die Zahlenfolge $a_0, a_1, a_2, \dots$ nach jedem Modul periodisch (d.h. entweder rein- oder gemischt-periodisch). Kommt vielleicht diese arithmetische Eigenschaft einer allgemeineren Funktionenklasse, z. B. den algebraischen Funktionen zu? *Nein*, besagt mein Beispiel; denn

$$1 + \binom{2}{1} x + \binom{4}{2} x^2 + \binom{6}{3} x^3 + \dots = \frac{1}{\sqrt{1-4x}}$$

ist eine algebraische Funktion.

Sur de certains systèmes d'équations différentielles,

par

PHILIPPE SIBIRANI, à Pavie, Italia.

1. Soit g une courbe gauche référée à un système d'axes $Oxyz$, P un point de g. Envisageons l'étoile de droites S et l'étoile de plans Σ, dont le centre est un point quelconque de l'espace, S engendrée par les droites parallèles aux rayons OP et Σ engendrée par les plans parallèles aux plans normaux à g en P. Soient $A(a, b, c)$, $B(\alpha, \beta, \gamma)$ deux points quelconques mais distincts et considérons les deux étoiles S' et Σ', la première de centre A et la seconde de centre B, référées projectivement à S et Σ respectivement. En précisant, si l'on prend O comme centre de S et de Σ, et si

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

sont les équations d'une droite r de S, les équations de la droite r' de S' correspondante soient :

$$\frac{x-a}{l'}=\frac{y-b}{m'}=\frac{z-c}{n'},$$

où l', m', n' sont liés à l, m, n par les équations :

$$\begin{aligned}
\rho l' &= a_{11}l+a_{12}m+a_{13}n \\
\rho m' &= a_{21}l+a_{22}m+a_{23}n \\
\rho n' &= a_{31}l+a_{32}m+a_{33}n
\end{aligned}$$

avec ρ coefficient numérique.

Si

$$\lambda x+\mu y+\nu z=0$$

est l'équation d'un plan π de Σ, l'équation du plan correspondant π' de Σ' soit

$$(x-\alpha)\lambda'+(y-\beta)\mu'+(z-\gamma)\nu'=0,$$

où λ', μ', ν' sont liés à λ, μ, ν par les équations :

$$\begin{aligned}
\tau \lambda' &= b_{11}\lambda+b_{12}\mu+b_{13}\nu \\
\tau \mu' &= b_{21}\lambda+b_{22}\mu+b_{23}\nu
\end{aligned}$$

$$\tau\,\nu' = b_{31}\lambda + b_{32}\mu + b_{33}\nu,$$

avec τ coefficient numérique.

Dans S' et Σ' nous considérerons correspondant une droite r' et un plan π', lorsque r' est la droite correspondante à la parallèle à OP et π' est le plan correspondant au plan parallèle ou plan normal en P à g.

Lorsque P se meut sur g, entre S' et Σ' naît une correspondance: appellons Γ la courbe lieu des intersections des couples d'éléments correspondants dans S' et Σ'.

Cela posé, nous démontrons que:

Les courbes Γ relatives aux ∞^2 solutions du système:

$$(1)\qquad \frac{dy}{dx} = \frac{P_2(x,y,z)}{P_1(x,y,z)}, \qquad \frac{dz}{dx} = \frac{P_3(x,y,z)}{P_1(x,y,z)},$$

où P_1, P_2, P_3 sont trois polynômes homogènes de degré k en x, y, z, appartiennent à une surface algébrique d'ordre $k+1$, qui passe par A et par B et qui a en A un point k-ple.

Si les coordonnés de P sont x, y, z, les équations de r' sont:

$$(2)\qquad \frac{X-a}{a_{11}x + a_{12}y + a_{13}z} = \frac{Y-b}{a_{21}x + a_{22}y + a_{22}z} = \frac{Z-c}{a_{31}x + a_{32}y + a_{33}z}$$

et l'équation de π', ayant regard à la (1), est:

$$(3)\quad (X-\alpha)(b_{11}P_1 + b_{12}P_2 + b_{13}P_3) + (Y-\beta)(b_{21}P_1 + b_{22}P_2 + b_{23}P_3) + (Z-\gamma)(b_{31}P_1 + b_{32}P_2 + b_{33}P_3) = 0.$$

En éliminant entre les (2) et (3) les rapports y/x, z/x, on obtient l'équation de la surface σ:

$$(4)\quad \begin{aligned} &(X-\alpha)\{b_{11}P_1(\xi,\eta,\zeta) + b_{12}P_2(\xi,\eta,\zeta) + b_{13}P_3(\xi,\eta,\zeta)\} \\ &+ (Y-\beta)\{b_{21}P_1(\xi,\eta,\zeta) + b_{22}P_2(\xi,\eta,\zeta) + b_{23}P_3(\xi,\eta,\zeta)\} \\ &+ (Z-\gamma)\{b_{31}P_1(\xi,\eta,\zeta) + b_{32}P_2(\xi,\eta,\zeta) + b_{33}P_3(\xi,\eta,\zeta)\} = 0, \end{aligned}$$

où pour abréger, nous avons posé:

$$\begin{aligned} \xi &= A_{11}(X-a) + A_{21}(Y-b) + A_{31}(Z-c) \\ \eta &= A_{12}(X-a) + A_{22}(Y-b) + A_{32}(Z-c) \\ \zeta &= A_{13}(X-a) + A_{23}(Y-b) + A_{33}(Z-c), \end{aligned}$$

et A_{ij} est le complément algébrique de a_{ij} dans

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

La σ contient évidemment toutes les courbes Γ relatives aux solutions du système (1), est algébrique d'ordre $k+1$, passe par A et B; et puisque le premier membre de (4) s'annule en $\xi=0, \eta=0, \zeta=0$ avec les dérivées partielles d'ordre $1, 2, \ldots. k-1$ rapport à ξ, η, ζ, la surface a en A un point k-ple.

2. À l'étoile S envisagée dans le § 1 on peut faire correspondre projectivement une étoile Σ_1 de plans de centre A, et à l'étoile Σ une étoile des droites S_1 de centre B. En précisant, si

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

sont les équations de r, soit

$$(x-a)\lambda'+(y-b)\mu'+(z-c)\nu'=0$$

l'équation du plan π' correspondant en Σ_1, où λ', μ', ν' sont liés à l, m, n par les équations:

$$\rho\lambda'=a_{11}l+a_{12}m+a_{13}n$$
$$\rho\mu'=a_{21}l+a_{22}m+a_{23}n$$
$$\rho\nu'=a_{31}l+a_{32}m+a_{33}n.$$

Et si

$$\lambda x+\mu y+\nu z=0$$

est l'équation d'un plan π de Σ, soient

$$\frac{x-\alpha}{l'}=\frac{y-\beta}{m'}=\frac{z-\gamma}{n'}$$

les équations de la droite r' correspondante en S_1, où l', m', n' sont liés à λ, μ, ν par les équations:

$$\tau l'=b_{11}\lambda+b_{12}\mu+b_{13}\nu$$
$$\tau m'=b_{21}\lambda+b_{22}\mu+b_{23}\nu$$
$$\tau n'=b_{31}\lambda+b_{32}\mu+b_{33}\nu.$$

Entre les droites de S_1 et les plans de Σ_1 on peut établir la correspondance analogue à celle que nous avons fait intercéder entre S' et Σ' au paragraphe précédent. Appelons Γ_1 la courbe lieu des intersections des couples des éléments correspondants dans S_1 et Σ_1.

Alors nous démontrons que: les courbes Γ_1 relatives aux ∞^3 solutions du système:

$$(6) \qquad x=P_1\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right), \qquad y=P_2\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

$$z = P_3\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

où P_1, P_2, P_3 sont des polynômes homogènes de degré k en $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ appartiennent à une surface algébrique d'ordre $k+1$, qui passe par A et B et qui a en B un point k-ple.

En ayant regard aux (6), l'équation des plan π' est:

$$(7)\quad (X-a)(a_{11}P_1 + a_{12}P_2 + a_{13}P_3) + (Y-b)(a_{21}P_1 + a_{22}P_2 + a_{23}P_3) + (Z-c)(a_{31}P_1 + a_{32}P_2 + a_{33}P_3) = 0$$

et les équations de la droites r' sont:

$$\frac{X-\alpha}{b_{11}x' + b_{12}y' + b_{13}z'} = \frac{Y-\beta}{b_{21}x' + b_{22}y' + b_{23}z'} = \frac{Z-\gamma}{b_{31}x' + b_{32}y' + b_{33}z'},$$

où nous avons posé, pour abréger:

$$x' = \frac{dx}{dt}, \qquad y' = \frac{dy}{dt}, \qquad z' = \frac{dz}{dt}.$$

En éliminant les rapports y'/x', z'/x' entre les (7) et les (8) nous avons l'équation de la surface τ_1:

$$\begin{aligned}(X-a)\{a_{11}P_1(\xi,\eta,\zeta) + a_{12}P_2(\xi,\eta,\zeta) + a_{13}P_3(\xi,\eta,\zeta)\}\\ + (Y-b)\{a_{21}P_1(\xi,\eta,\zeta) + a_{22}P_2(\xi,\eta,\zeta) + a_{23}P_3(\xi,\eta,\zeta)\}\\ + (Z-c)\{a_{31}P_1(\xi,\eta,\zeta) + a_{32}P_1(\xi,\eta,\zeta) + a_{33}P_3(\xi,\eta,\zeta)\} = 0,\end{aligned}$$

où nous avons posé:

$$\begin{aligned}\xi &= B_{11}(X-\alpha) + B_{21}(Y-\beta) + B_{31}(Z-\gamma)\\ \eta &= B_{12}(X-\alpha) + B_{22}(Y-\beta) + B_{32}(Z-\gamma)\\ \zeta &= B_{13}(X-\alpha) + B_{23}(Y-\beta) + B_{33}(Z-\gamma)\end{aligned}$$

et B_{ij} est le complément algébrique de b_{ij} dans le déterminant

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}.$$

La σ_1 contient évidemment toutes les courbes Γ_1 relatives aux solutions du système (6); elle est algébrique d'ordre $k+1$, passe par les points A et B et a en B un point k-ple.

3. L'étoile Σ envisagée au §1 soit remplacée par l'étoile engendrée par les plans parallèles aux plans osculateurs à γ. Alors nous démontrons que: les courbes Γ relatives aux ∞^2 solutions du système (1) appartiennent à une surface algébrique d'ordre $3k$ qui passe par les points A et B et

qui a en A un point $(3k-1)$-ple.

D'après la formule (1) on obtient :

$$\frac{d^2y}{dx^2}=\frac{P_1\left[P_1\frac{\partial P_2}{\partial x}+P_2\frac{\partial P_2}{\partial y}+P_3\frac{\partial P_2}{\partial z}\right]-P_2\left[P_1\frac{\partial P_1}{\partial x}+P_2\frac{\partial P_1}{\partial y}+P_3\frac{\partial P_1}{\partial z}\right]}{P_1^3},$$

$$\frac{d^2z}{dx^2}=\frac{P_1\left[P_1\frac{\partial P_3}{\partial x}+P_2\frac{\partial P_3}{\partial y}+P_3\frac{\partial P_3}{\partial z}\right]-P_3\left[P_1\frac{\partial P_1}{\partial x}+P_2\frac{\partial P_1}{\partial y}+P_3\frac{\partial P_1}{\partial z}\right]}{P_1^3}.$$

Alors des nombres proportionnels aux cosinus directeurs de la binormale aux courbes γ solutions du système (1) sont :

$$H_1=P_2\left[P_1\frac{\partial P_3}{\partial x}+P_2\frac{\partial P_3}{\partial y}+P_3\frac{\partial P_3}{\partial z}\right]-P_3\left[P_1\frac{\partial P_2}{\partial x}+P_2\frac{\partial P_2}{\partial y}+P_3\frac{\partial P_2}{\partial z}\right]$$

$$H_2=P_3\left[P_1\frac{\partial P_1}{\partial x}+P_2\frac{\partial P_1}{\partial y}+P_3\frac{\partial P_1}{\partial z}\right]-P_1\left[P_1\frac{\partial P_3}{\partial x}+P_2\frac{\partial P_3}{\partial y}+P_3\frac{\partial P_3}{\partial z}\right]$$

$$H_3=P_1\left[P_1\frac{\partial P_2}{\partial x}+P_2\frac{\partial P_2}{\partial y}+P_3\frac{\partial P_2}{\partial z}\right]-P_2\left[P_1\frac{\partial P_1}{\partial x}+P_2\frac{\partial P_1}{\partial y}+P_3\frac{\partial P_1}{\partial z}\right]$$

polynômes homogènes en x, y, z de degré $3k-1$.

L'équation de π' devient

$$(9)\quad (X-\alpha)\{b_{11}H_1+b_{12}H_2+b_{13}H_3\}+(Y-\beta)\{b_{21}H_1+b_{22}H_2+b_{23}H_3\}$$
$$+(Z-\gamma)\{b_{31}H_1+b_{32}H_2+b_{33}H_3\}=0.$$

En éliminant les rapports y/x, z/x entre les (2) et (9) nous avons l'équation de la surface σ_2 :

$$(X-\alpha)\{b_{11}H_1(\xi,\eta,\zeta)+b_{12}H_2(\xi,\eta,\zeta)+b_{13}H_3(\xi,\eta,\zeta)\}$$
$$+(Y-\beta)\{b_{21}H_1(\xi,\eta,\zeta)+b_{22}H_2(\xi,\eta,\zeta)+6_{23}H_3(\xi,\eta,\zeta)\}$$
$$+(Z-\gamma)\{b_{31}H_1(\xi,\eta,\zeta)+b_{32}H_2(\xi,\eta,\zeta)+b_{33}H_3(\xi,\eta,\zeta)\}=0,$$

où ξ, η, ζ sont donnés par les (5).

La surface σ_2 est algébrique d'ordre $3k$, passe par les points A et B, et a en A un point $(3k-1)$-ple ; elle contient évidemment toutes les courbes Γ qui sont relatives aux solutions du système (1).

Über die Mittelwerte analytischer Funktionen,

von

G. Szegö in Berlin.

In der vorliegenden Arbeit beweise ich das folgende Theorem:

Es seien

$$C_0, C_1, C_2, \ldots, C_k, \ldots$$

geschlossene, doppelpunktlose, stetige und rektifizierbare Kurven in der komplexen z-Ebene, deren jede in der nachfolgenden enthalten ist und die gleichmässig gegen den Einheitskreis $|z|=1$ *konvergieren*(1). *Die Länge von* C_k *sei* l_k *und es sei*

$$\lim_{k=\infty} l_k = 2\pi.$$

Es sei ferner

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

eine analytische Funktion, die im Innern und auf C_k $(k=0, 1, 2, \ldots)$ *regulär ist und für welche*

$$\frac{1}{l_k}\int_{C_k} |f(z)|^2 \, ds < M \qquad (k=0, 1, 2, \ldots)$$

gilt; ds bezeichnet hierbei das Bogenelement von C_k *und* M *eine von* k *unabhängige Zahl. Dann ist*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \leqq M$$

für alle $r < 1$.

Aus diesem Theorem ergibt sich fast unmittelbar die folgende Erweiterung eines Fatouschen Satzes(2):

Es sei $f(z)$ *regulär-analytisch im Innern des Einheitskreises,* $0 \leqq \alpha < \beta \leqq 2\pi$ *feste Zahlen. Es sei ferner*

$$\int_\alpha^\beta |f(re^{i\theta})|^2 \, d\theta$$

beschränkt für $r<1$. *Dann existiert*

---

(1) Vgl. § 1.

(2) Vgl. a. a. O. Man siehe Fussnote (1) in der Seite 91 dieses Heftes.

$$\lim_{r=1} f(re^{i\theta}) = f(e^{i\theta})$$

für $\alpha \leqq \theta \leqq \beta$ (mit eventueller Ausnahme einer Menge vom Lebesgueschen Masse 0) *und die Funktion* $|f(e^{i\theta})|^2$ *ist im Lebesgueschen Sinne integrierbar für* $\alpha \leqq \theta \leqq \beta$.

Für $\alpha = 0$, $\beta = 2\pi$ erhält man hieraus den Satz von Fatou.

Das wesentliche Hilfsmittel zum Beweis dieser Theoreme bilden gewisse Polynome, welche einer beliebigen Kurve zugeordnet sind und welche ich in einer früheren Arbeit(1) definiert und eingehend untersucht habe. Nachdem in § 1 Hilfssätze über Kurvenfolgen vorausgeschickt werden, stelle ich in § 2 diejenigen Eigenschaften dieser Polynome zusammen, die im folgenden benötigt werden. In §§ 3–5 folgen die Beweise der oben formulierten Theoreme.

§ 1.

Über reguläre Kurvenfolgen.

Unter einer *regulären Kurvenfolge* (C_k) verstehe ich eine Folge von Kurven

$$C_0,\ C_1,\ C_2, \ldots, C_k, \ldots$$

mit folgenden Eigenschaften:

a) Sie sind geschlossen, doppelpunktlos, stetig und rektifizierbar.

b) Jede Kurve C_k ist in der nachfolgenden C_{k+1} (im Innern oder am Rande derselben) enthalten.

c) Bezeichnet ε eine beliebig kleine positive Zahl, dann ist für $k > K(\varepsilon)$ jede Kurve C_k im Kreisring

$$1 - \varepsilon \leqq |z| \leqq 1$$

enthalten(2).

d) Für die Länge l_k von C_k gilt die Gleichung

$$\lim_{k=\infty} C_k = 2\pi.$$

Ich beweise zunächst den

Hilfssatz I.—*Es sei* (C_k) *eine reguläre Kurvenfolge. Dann ist*

$$\lim_{k=\infty} \int_{C_k} \left| 1 - \frac{dz}{iz\, ds} \right|^2 ds = 0.$$

Das hier auftretende Integral ist als Grenzwert des Ausdruckes

(1) A. a. O. Siehe Fussnote (1) in der Seite 91 dieses Heftes.

(2) Für genügend grosse k enthält C_k sicherlich den Nullpunkt im Innern. Wir wollen der Einfachheit halber annehmen, dass das von Anfang an der Fall ist.

$$\sum_{\nu=1}^{m}\left|1-\frac{\operatorname{sgn}(z_\nu^{(m)}-z_{\nu-1}^{(m)})}{i\zeta_\nu^{(m)}}\right|^2 |z_\nu^{(m)}-z_{\nu-1}^{(m)}|$$

definiert, wobei $z_0^{(m)}, z_1^{(m)}, \ldots, z_{m-1}^{(m)}, z_m^{(m)}=z_0^{(m)}$ irgend eine Einteilung von C_k (in der Reihenfolge des positiven Umlaufes), $\zeta_\nu^{(m)}$ eine beliebige Stelle auf C_k zwischen $z_{\nu-1}^{(m)}$ und $z_\nu^{(m)}$ bezeichnet und

$$\lim_{m=\infty}\ \operatorname*{Max}_{1<\nu\leq m}\left|z_\nu^{(m)}-z_{\nu-1}^{(m)}\right|=0$$

ist.

Man hat nach einem geläufigen Satz

$$2\pi=\int_{C_k}\frac{\cos\delta\, ds}{|z|},$$

wobei δ den Winkel zwischen dem Radiusvektor und der äusseren Normale bezeichnet. Nun ist

$$\frac{dz}{izds}=\frac{1}{|z|}e^{i\delta},$$

d. h.

$$\left|1-\frac{dz}{izds}\right|^2=1+\frac{1}{|z|^2}-2\frac{\cos\delta}{|z|}.$$

Daraus folgt, wenn l_k die Länge von C_k bezeichnet,

$$\int_{C_k}\left|1-\frac{dz}{izds}\right|^2 ds=l_k+\int_{C_k}\frac{ds}{|z|^2}-4\pi,$$

woraus wegen c) und d) die Behauptung klar ist.

Hilfssatz II. *Es sei* (C_k) *eine reguläre Kurvenfolge. Dann ist*

$$\lim_{k=\infty}\int_{C_k} z^m\,\bar z^n\, ds=2\pi e_{mn}\qquad (m, n=0, 1, 2, \ldots),$$

wobei $e_{mn}=0$ *oder* 1, *je nachdem* $m\gtrless n$ *oder* $m=n$ *ist*(1).

Allgemeiner beweise ich die Gleichung

$$\lim_{k=\infty}\int_{C_k} h(z)\,\bar z^n\, ds=\int_0^{2\pi} h(e^{i\theta})e^{-in\theta}\,d\theta,$$

wobei n eine feste (nichtnegative) ganze Zahl, $h(z)$ eine für $|z|\leqq 1$ reguläre-analytische Funktion bezeichnet. Es sei $|h(z)|<H$ für $|z|\leqq 1$. Nach dem Cauchyschen Satz ist das Integral rechterhand gleich

$$\frac{1}{i}\int_{|z|=1}\frac{h(z)}{z^{n+1}}\,dz=\frac{1}{i}\int_{C_k}\frac{h(z)}{z^{n+1}}\,dz\ (^2).$$

(1) $\bar z$ bezeichnet die zu z konjugiert-komplexe Zahl.

(2) Vgl. die Fussnote (1), Seite 91 dieses Heftes.

Es handelt sich also um den Beweis der Gleichung

$$\lim_{k=\infty} \int_{C_k} h(z)\left(\bar{z}^n - \frac{1}{iz^{n+1}} \frac{dz}{ds}\right) ds = 0 .$$

Es sei ε eine beliebig kleine positive Zahl. Wegen c) gilt für $k > K(\varepsilon)$

$$\left| \int_{C_k} h(z)\left(\bar{z}^n - \frac{1}{iz^{n+1}} \frac{dz}{ds}\right) ds \right| < \frac{H}{(1-\varepsilon)^n} \int_{C_k} \left| |z|^{2n} - \frac{1}{iz} \frac{dz}{ds} \right| ds$$

$$< \frac{Hl_k}{(1-\varepsilon)^n}\left[1-(1-\varepsilon)^{2n}\right] + \frac{H}{(1-\varepsilon)^n} \int_{C_k} \left| 1 - \frac{1}{iz} \frac{dz}{ds} \right| ds .$$

Das erste Glied wird hier mit ε beliebig klein, für das zweite können wir die Schwarzsche Ungleichung anwenden. Es ist

$$\left(\int_{C_k} \left| 1 - \frac{1}{iz} \frac{dz}{ds} \right| ds \right)^2 \leqq l_k \int_{C_k} \left| 1 - \frac{1}{iz} \frac{dz}{ds} \right|^2 ds ,$$

woraus wegen Hilfssatz I die Behauptung folgt.

§ 2.

Über die orthogonalen Polynome, die zu einer gegebenen Kurve gehören.

1. Es sei C eine geschlossene, doppelpunktlose, stetige und rektifizierbare Kurve mit der Länge l. Man betrachte die Funktionen

$$1,\ z,\ z^2, \ldots ,\ z^n, \ldots$$

und orthogonalisiere dieselben nach dem üblichen Verfahren, unter Zugrundelegung einer besonderen Integrationsvorschrift. Man bilde namentlich Funktionen von der Form

$$\begin{aligned} &P_0(z) = c_{00} , \\ &P_1(z) = c_{10} + c_{11} z , \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &P_n(z) = c_{n0} + c_{n1} z + \ldots + c_{nn} z^n , \\ &\ldots\ldots\ldots\ldots\ldots\ldots , \end{aligned}$$

für welche

$$\frac{1}{l} \int_C P_m(z) \overline{P_n(z)}\, ds = \begin{cases} 0 \text{ für } m \gtrless n , \\ 1 \text{ für } m = n \quad (m, n = 0, 1, 2, \ldots) \end{cases}$$

gilt. Die Polynome

$$P_0(z),\ P_1(z), \ldots ,\ P_n(z), \ldots$$

bilden dann ein *normiertes Orthogonalsystem.* Setzt man noch

$$c_{nn} > 0 \qquad (n = 0, 1, 2, \ldots)$$

voraus, dann sind die durch C eindeutig bestimmt. Sie werden als die *zu der Kurve C gehörigen orthogonalen Polynome* bezeichnet([1]).

2. Nach den klassischen Sätzen über Orthogonalisierung lassen sich diese Polynome folgendermassen darstellen: Es sei

$$g_{pq}=\frac{1}{l}\int_C z^p \bar{z}^q \, ds \qquad (p, q=0, 1, 2, \ldots.)$$

und man bezeichne mit D_n die Determinante

$$D_n=[g_{pq}]_0^n \qquad (n=0, 1, 2, \ldots.) \ ([2]).$$

Dann ist

$$P_0(z)=1\,; \quad P_n(z)=\frac{1}{\sqrt{D_{n-1}D_n}}\begin{vmatrix} g_{00} & g_{10} & \cdots & g_{n0} \\ g_{01} & g_{11} & \cdots & g_{n1} \\ \cdots & \cdots & \cdots & \cdots \\ g_{0,\,n-1} & g_{1,\,n-1} & \cdots & g_{n,\,n-1} \\ 1 & z & \cdots & z^n \end{vmatrix} \quad (n=1,2,3, \ldots.)([3]).$$

3. Ich führe ferner die folgende Formel an, die in der Theorie der orthogonalen Funktionen oft als Parsevalsche Formel erwähnt wird:

Es sei $f(z)$ regulär-analytisch im Innern und auf C und sei

$$\frac{1}{l}\int_C f(z)\overline{P_n(z)}\, ds=c_n \qquad (n=0, 1, 2, \ldots\ ).$$

Dann ist

$$|c_0|^2+|c_1|^2+\ldots.+|c_n|^2+\ldots.=\frac{1}{l}\int_C |f(z)|^2 ds \ ([4]).$$

4. Ich schliesse diesen §-en mit

Hilfssatz III. *Es sei (C_k) eine reguläre Kurvenfolge und $P_n^{(k)}(z)$ $(n=0, 1, 2, \ldots.)$ seien die zur Kurve C_k gehörigen orthogonalen Polynome $(k=0, 1, 2, \ldots.)$. Es ist bei festem n*

$$\lim_{k=\infty} P_n^{(k)}(z)=z^n \qquad (n=0, 1, 2, \ldots.).$$

Wir setzen

$$g_{pq}^{(k)}=\frac{1}{l_k}\int_{C_k} z^p \bar{z}^q ds, \quad D_n^{(k)}=[g_{pq}^{(k)}]_0^n.$$

([1]) Vgl. meine Arbeit: Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören [Mathematische Zeitschrift Bd. 9 (1921), S. 218–270].

([2]) Sämtliche D_n sind positiv.

([3]) Sämtliche Quadratwurzeln sind positiv zu nehmen.—Vgl. a. a. O. S. 227–228.

([4]) Vgl. a. a. O. S. 232–233. Man siehe Fussnote (1) oben.

Dann ist nach Hiltssatz II

$$\lim_{k=\infty} g_{pq}^{(k)} = c_{pq}\,,$$

d. h.

$$\lim_{k=\infty} D_n^{(k)} = [c_{pq}]_0^n = 1$$

und

$$\lim_{k=\infty} \begin{vmatrix} g_{00}^{(k)} & g_{10}^{(k)} & \dots & g_{n0}^{(k)} \\ g_{01}^{(k)} & g_{11}^{(k)} & \dots & g_{n1}^{(k)} \\ \dots & \dots & \dots & \dots \\ g_{0,\,n-1}^{(k)} & g_{1,\,n-1}^{(k)} & \dots & g_{n,\,n-1}^{(k)} \\ 1 & z & \dots & z^n \end{vmatrix} = \begin{vmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & \dots & & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 1 & z & \dots & z^{n-1} & z^n \end{vmatrix} = z^n\,.$$

Daraus folgt die Behauptung.

§ 3.

Ein Satz über die Mittelwerte analytischer Funktionen.

Es sei (C_k) *eine reguläre Kurvenfolge und*

$$f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots$$

eine analytische Funktion, die im Innern und auf den Kurven C_k *regulär ist*(1). *Es sei ferner*

$$\frac{1}{l_k} \int_{C_k} |f(z)|^2 \, ds < M \qquad (k = 0, 1, 2, \dots),$$

wobei l_k *die Länge von* C_k *bezeichnet. Dann ist*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \leqq M \qquad (r < 1).$$

Vorbemerkung. Es ist

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = |a_0|^2 + |a_1|^2 r^2 + \dots + |a_n|^2 r^{2n} + \dots,$$

so dass die letzte Ungleichung mit der Konvergenz der Reihe

$$a = |a_0|^2 + |a_1|^2 + \dots + |a_n|^2 + \dots$$

(bzw. mit der Ungleichung $a \leqq M$) gleichbedeutend ist.

Beweis. Wir bezeichnen—wie in § 2—mit $P_n^{(k)}(z)$ die zu C_k gehörigen orthogonalen Polynome. Wir setzen ferner

$$a_n^{(k)} = \frac{1}{l_k} \int_{C_k} f(z) \overline{P_n^{(k)}(z)} \, ds \qquad (n = 0, 1, 2, \dots; \; k = 0, 1, 2, \dots).$$

---

(1) Vgl. die Bemerkung in Fussnote (2) der vorigen Seite.

Dann ist

$$|a_0^{(k)}|^2+|a_1^{(k)}|^2+\ldots+|a_n^{(k)}|^2+\ldots=\frac{1}{l_k}\int_{C_k}|f(z)|^2\,ds<M.$$

Man hat somit bei festem m

$$|a_0^{(k)}|^2+|a_1^{(k)}|^2+\ldots+|a_m^{(k)}|^2<M.$$

Ich behaupte nun, dass bei festem n die Gleichung

$$\lim_{k=\infty} a_n^{(k)}=a_n \tag{1}$$

gilt. Wird dies gezeigt, so folgt aus der letzten Ungleichung

$$|a_0|^2+|a_1|^2+\ldots+|a_m|^2\leqq M$$

für jedes m und hieraus die Behauptung.

Alles geht somit darauf hinaus, (1) zu beweisen. Es ist nach dem Satz von Cauchy (für genügend grosse k)

$$a_n=\frac{1}{2\pi i}\int\limits_{C_k}\frac{f(z)}{z^{n+1}}\,dz,$$

d. h.

$$a_n-a_n^{(k)}=\frac{1}{l_k}\int_{C_k}f(z)\left(\frac{l_k}{2\pi i z^{n+1}}\,\frac{dz}{ds}-\overline{P_n^{(k)}(z)}\right)ds.$$

Hieraus folgt mit Hilfe der Schwarzschen Ungleichung

$$|a_n-a_n^{(k)}|^2\leqq\frac{1}{l_k}\int_{C_k}|f(z)|^2\,ds\,.\,\frac{1}{l_k}\int_{C_k}\left|\frac{l_k}{2\pi i z^{n+1}}\,\frac{dz}{ds}-\overline{P_n^{(k)}(z)}\right|^2 ds$$

$$<M\,.\,\frac{1}{l_k}\int\limits_{C_k}\left|\frac{l_k}{2\pi i\, z^{n+1}}\,\frac{dz}{ds}-\overline{P_n^{(k)}(z)}\right|^2 ds.$$

Mit Rücksicht auf die Hilfssätze I und III ergibt sich hieraus (1). Damit ist der eingangs formulierte Satz bewiesen.

§ 4.

Über eine Verallgemeinerung des Fatouschen Satzes.

Erster Teil des Beweises.

Herr Fatou hat im Jahre 1906 den folgenden wichtigen Satz bewiesen:

Es sei

$$f(z)=a_0+a_1z+\ldots+a_nz^n+\ldots$$

eine Potenzreihe, für welche

$$|a_0|^2+|a_1|^2+\ldots+|a_n|^2+\ldots$$

konvergiert, die also für $|z|<1$ *eine reguläre analytische Funktion darstellt. Dann existiert mit eventueller Ausnahme einer Menge vom Lebesgueschen Masse* 0 *der Grenzwert*

$$\lim_{r=1} f(re^{i\theta}) = f(e^{i\theta})$$

und die Funktion $|f(e^{i\theta})|^2$ *ist im Lebesgueschen Sinne integrabel für*

$$0 \leqq \theta \leqq 2\pi \text{ (1)}.$$

Ich will nun auf Grund des in § 3 bewiesenen Satzes das folgende Theorem herleiten:

Es sei $f(z)$ *eine im Innern des Einheitskreises reguläre analytische Funktion und* $0 \leqq \alpha < \beta \leqq 2\pi$ *feste Zahlen. Es sei ferner für alle* $r<1$

$$\int_\alpha^\beta |f(re^{i\theta})|^2 \, d\theta < M.$$

Dann existiert

$$\lim_{r=1} f(re^{i\theta}) = f(e^{i\theta}) \tag{2}$$

für $\alpha \leqq \theta \leqq \beta$, *mit eventueller Ausnahme einer Menge vom Lebesgueschen Masse* 0. *Die Funktion* $|f(e^{i\theta})|^2$ *ist im Lebesgueschen Sinne integrierbar für* $\alpha \leqq \theta \leqq \beta$.

1. Es genügt offenbar die Gleichung (2) für das Intervall $\alpha+\varepsilon \leqq \theta \leqq \beta-\varepsilon$ zu beweisen(²), wobei ε eine beliebig kleine, aber feste positive Zahl bezeichnet $\left(\varepsilon < \frac{\beta-\alpha}{2}\right)$. Ich behaupte zunächst die Existenz von zwei Zahlen θ_1, θ_2, die bzw. die Ungleichungen

$$\alpha \leqq \theta_1 \leqq \alpha+\varepsilon, \qquad \beta-\varepsilon \leqq \theta_2 \leqq \beta$$

erfüllen, für die ferner die Integrale

$$\int_0^{r_1} |f(re^{i\theta_1})|^2 \, dr, \qquad \int_0^{r_2} |f(re^{i\theta_2})|^2 \, dr$$

bei beliebigen $r_1 < 1$ bzw. $r_2 < 1$ unterhalb einer von r_1 und r_2 unabhängigen oberen Schranke M' liegen. Sowohl M' wie auch θ_1 und θ_2 hängen hierbei von ε ab.

Es sei für $R<1$, $\alpha \leqq \theta \leqq \beta$

$$J(\theta, R) = \int_0^R |f(re^{i\theta})|^2 \, dr$$

gesetzt, dann ist laut Voraussetzung

(¹) Séries trigonométriques et séries de Taylor [Acta Mathematica, Bd. 30 (1906) S. 335–400], S. 377–379.

(²) Natürlich wiederum unter der Zulassung einer Ausnahmemenge vom Masse 0.

$$\int_\alpha^{\alpha+\varepsilon} J(\theta, R)\, d\theta \leqq \int_\alpha^\beta J(\theta, R)\, d\theta \leqq RM < M.$$

Es gehört also zu jedem $R<1$ mindestens eine Stelle $\theta_1(R)$ $(\alpha \leqq \theta_1(R) \leqq \alpha+\varepsilon)$, für die

$$J(\theta_1(R), R) = \int_0^R |f(re^{i\theta_1(R)})|^2\, dr < \frac{M}{\varepsilon}.$$

Wir wählen eine monotone Folge R_ν von R-Werten, für die

$$\lim_{\nu=\infty} R_\nu = 1$$

ist und

$$\lim_{\nu=\infty} \theta_1(R_\nu) = \theta_1$$

existiert $(\alpha \leqq \theta_1 \leqq \alpha+\varepsilon)$. Dann ist bei festem $0 \leqq R < 1$

$$\int_0^R |f(re^{i\theta_1})|^2\, dr = \lim_{\nu=\infty} \int_0^R |f(re^{i\theta_1(R_\nu)})|^2\, dr$$

und da für genügend grosse ν stets $R_\nu > R$ ist, so ist

$$\int_0^R \left|f(re^{i\theta_1(R_\nu)})\right|^2 dr \leqq \int_0^{R_\nu} \left|f(re^{i\theta_1(R_\nu)})\right|^2 dr < \frac{M}{\varepsilon},$$

d. h.

$$\int_0^R |f(re^{i\theta_1})|^2\, dr \leqq \frac{M}{\varepsilon}.$$

Ähnlich zeigt man die Existenz von θ_2. Wir können also $M' = \frac{M}{\varepsilon}$ setzen.

2. Wir bilden mit Hilfe der Funktion

(3) $$z = p(\zeta)$$

den Kreissektors S

$$0 \leqq |z| < 1, \qquad \theta_1 \leqq \operatorname{arc} z \leqq \theta_2$$

konform und schlicht auf das Innere des Einheitskreises der ζ-Ebene ab derart, dass etwa der Punkt $z = \frac{1}{2} e^{i\frac{\alpha+\beta}{2}}$ in $\zeta = 0$ übergehe und dass $p'(0) = 1$ sei. Es bezeichne G den Kreisbogen $z = e^{i\theta}$ $(\theta_1 \leqq \theta \leqq \theta_2)$ und Γ das Bild $\zeta = e^{i\varphi}$ $(\varphi_1 \leqq \varphi \leqq \varphi_2)$ von G. Die Funktion (3) ist dann noch analytisch und schlicht auf Γ (mit Ausschluss der Endpunkte), sogar auch in einem genügend kleinen Bereiche, der den Bogen $\varphi_1 + \eta \leqq \varphi \leqq \varphi_2 - \eta$ ganz im Innern enthält, wobei η eine beliebig kleine positive Zahl bezeichnet.

Es sei L_r die Kurve, die aus den beiden Strecken

$$0 \leqq |z| \leqq r, \qquad \operatorname{arc} z = \theta_1 \text{ bzw. } \theta_2$$

und aus dem Kreisbogen

$$|z| = r, \qquad \theta_1 \leqq \operatorname{arc} z \leqq \theta_2$$

besteht und Λ_r ihr Bild in der ζ-Ebene. Man sieht unmittelbar ein, dass wenn r_k $(k=0, 1, 2, \ldots.)$ irgend eine monoton gegen 1 zunehmende Folge ist, die Kurven $\Lambda_{r_k} (k=0, 1, 2, \ldots.)$ im Sinne von §1 eine reguläre Kurvenfolge bilden.

3. Die Funktion

$$F(\zeta) = f[p(\zeta)] \sqrt{p'(\zeta)}$$

ist regulär-analytisch für $|\zeta| < 1$, sogar auch für $|\zeta| = 1$, mit Ausnahme der drei Punkte, die $z=0$, $z=e^{i\theta_1}$, $z=e^{i\theta_2}$ entsprechen. (Hierbei ist unter $\sqrt{p'(\zeta)}$ derjenige Zweig zu verstehen, der für $\zeta=0$ gleich 1 ist; wegen der Schlichtheit ist überall $p'(\zeta) \neq 0$.) Insbesondere ist sie regulär auf Λ_r für jedes $r < 1$. Man hat ferner

$$\int_{\Lambda r} |F(\zeta)|^2 d\sigma = \int_{L_r} |f(z)|^2 ds < 2M' + M = 2\frac{M}{\varepsilon} + M = M'',$$

wobei ds das Bogenelement von L_r, $d\sigma = |p'(\zeta)|^{-1} ds$ das von Λ_r bezeichnet. Man schliesst hieraus wegen § 3 für $\rho < 1$

$$\frac{1}{2\pi} \int_0^{2\pi} |F(\rho e^{i\varphi})|^2 d\varphi \leqq M'',$$

woraus nach Fatou (mit Ausnahme einer 0-Menge von φ-Werten) die Existenz des Grenzwertes

$$\lim_{\rho=1} F(\rho e^{i\varphi}) = F(e^{i\varphi})$$

für $\varphi_1 \leqq \varphi \leqq \varphi_2$ folgt. Da auf Γ (mit Ausschluss der Endpunkte) $\sqrt{p'(\zeta)}$ regulär und von 0 verschieden ist, folgt hieraus

$$\lim_{\rho=1} f[p(\rho e^{i\varphi})] = f[p(e^{i\varphi})] = f(e^{i\theta}). \tag{4}$$

§ 5.

Über eine Verallgemeinerung des Fatouschen Satzes.

Zweiter Teil (Schluss) des Beweises.

Auf (4) folgt noch nicht unmittelbar (2), da bei der Abbildung (3) die Radien $re^{i\theta}$ nicht den Radien $\rho e^{i\varphi}$, sondern gewissen Kurven $P(\varphi)$, die durch den Punkt $e^{i\varphi}$ hindurchgehen und auf Γ orthogonal sind, ents-

prechen. Um also (2) zu erhalten, müssen wir zeigen, dass (bei festem φ) die Gleichung

$$(4') \qquad \lim F(\zeta)=F(e^{i\varphi})$$

folgt, wobei die Annäherung an der Kurve $P(\varphi)$ gemeint ist.

Wir bezeichnen mit $\zeta_1=\zeta_1(\rho)$, $\zeta_2=\zeta_2(\rho)$ den Schnittpunkt des Kreises $|\zeta|=\rho$ mit dem Radius $0e^{i\varphi}$ bzw. mit der Kurve $P(\varphi)$. Für genügend kleine Werte von $1-\rho$ sind diese Punkte gewiss eindeutig bestimmt. Die Gleichung (4) erhält alsdann die Form

$$\lim_{\rho=1} F[\zeta_1(\rho)]=F(e^{i\varphi})$$

und die zu beweisende Gleichung (4′) die Form

$$\lim_{\rho=1} F[\zeta_2(\rho)]=F(e^{i\varphi}).$$

Es genügt ferner zu zeigen, dass der Ausdruck

$$F[\zeta_1(\rho)]-F[\zeta_2(\rho)]$$

für $\lim \rho=1$ gegen 0 strebt.

Da die analytische Kurve $P(\varphi)$ von dem Radius $0e^{i\varphi}$ im Punkte $e^{i\varphi}$ berührt wird, so ist ersichtlich

$$|\zeta_1(\rho)-\zeta_2(\rho)|<c(1-\rho)^2,$$

wobei c von ρ unabhängig ist. Es sei

$$F(\zeta)=\sum_{n=0}^{\infty} A_n \zeta^n,$$

dann ist nach den Vorhergehenden

$$|A_0|^2+|A_1|^2+\dots+|A_n|^2+\dots\leqq M',$$

ferner

$$|F(\zeta_1)-F(\zeta_2)|\leqq|\zeta_1-\zeta_2|\sum_{n=1}^{\infty}|A_n|(|\zeta_1|^{n-1}+|\zeta_1|^{n-1}|\zeta_2|+\dots+|\zeta_2|^{n-1})$$

$$\leqq c(1-\rho)^2\sum_{n=1}^{\infty} n|A_n|\rho^{n-1}.$$

Mit Hilfe der Schwarzschen Ungleichung schliesst man hieraus

$$|F(\zeta_1)-F(\zeta_2)|\leqq c(1-\rho)^2\sqrt{M'}\sqrt{\sum_{n=1}^{\infty} n^2\rho^{2n-2}}$$

$$=c\sqrt{M'}\frac{(1-\rho)^2(1+\rho^2)^{\frac{1}{2}}}{(1-\rho^2)^{\frac{3}{2}}},$$

d. h.

$$|F(\zeta_1)-F(\zeta_2)| < c'(1-\rho)^{\frac{1}{2}},$$

wobei c' von ρ unabhängig ist. Daraus folgt die Behauptung.

Bei der Abbildung (3) geht ein Intervall von φ-Werten von der Länge l, das im Intervalle $\varphi_1+\eta \leqq \varphi \leqq \varphi_2-\eta$ liegt (η ist positiv), in ein solches von θ-Werten über dessen Länge höchstens Nl ist, wobei N eine feste (nur von η abhängende) Zahl ist. Daraus folgt, dass eine 0-Menge von φ-Werten in eine eben solche Menge von θ-Werten übergeführt ist, d.h. (2) gilt auch höchstens mit Ausnahme einer 0-Menge.

Die Integrierbarkeit von $|f(e^{i\theta})|^2$ für $\alpha \leqq \theta \leqq \beta$ folgt unmittelbar aus einem Lemma von Fatou(1).

Berlin, Januar 1921.

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(1) A. a. O. S. 375. Man siehe Fussnote in der Seite 94.

Axiomatic Investigation on Number-Systems, I,

by

KUNIZO YONEYAMA, Fukuoka.

Contents.

Introduction.

Historically, the conception of number was gradually enlarged from that of natural number to those of fractional, real, negative and complex numbers; and during the time in which this enlargement was going on, very many different methods of introducing these numbers were proposed

by many mathematicians. Some of them are axiomatic, and some of them are synthetic while the others are analytic. Some are scientific while the others are popular. Therefore it is most natural that, when these methods are used to extend the conception of number, step by step, they lack unification. Among them, there are two analytic methods commonly used in a scientific treatment of this kind. The first is rather older than the second and is called the algebraic method by L. Couturat. It introduces a new number on the principle that the mathematical operations must be performed without any restriction. For example, a negative number is introduced in order to perform subtraction without any restriction, and a fraction to perform division in a similar manner. But it was pointed out by some mathematicians that this method contains a logical error in itself, strictly speaking. The second is that which is called the arithmetical method by the above mathematician, and that which introduces a new number by using a pair of numbers already known. This method was invented by W. R. Hamilton to introduce the ordinary complex number; and then it was used by C. Weierstrass to introduce the relative number from the absolute number, and lastly by J. Tannery to introduce the absolute rational number from the natural number. But this method was tried to introduce the above three numbers only, as far as I know.

Here, by the last method, I shall try to form the systematic developement of whole number-system, such that the whole structure of number-system has unification and rigorousness as much as possible. Namely, *from only one thing and two fundamental operations, I introduce two fundamental numbers by most natural consideration; and starting with these two fundamental numbers, I introduce all natural numbers by pair of numbers already defined. Next I introduce whole integers (positive and negative natural numbers and zero) by pairs of natural numbers already defined; and similarly I introduce whole rational numbers by pairs of integers already defined; and whole real numbers by pairs of derived rational numbers (which are derived at once from rational numbers already defined). Further, I introduce whole complex numbers by pairs of real numbers already defined; and lastly I introduce whole quaternions by pairs of complex numbers already defined. Thus whole number-system commencing from natural numbers and ending with quaternions is formed, step by step, by quite the same method and in a very rigorous manner.* The caution has also been taken to give a similar form to the postulates and definitions when every new number is introduced, so that the whole system may have

unification as much as possible. All these are discussed in Chapter I.

Next to introduce the number-system by the axiomatic method, E. V. Huntington used ten fundamental laws of algebra as a set of postulates, taking two operations (addition and multiplication) as undefined terms. To this set of postulates, he added certain postulates of existence, and thus he obtained several different number-systems according as different postulates of existence are added. This method is very interesting. But, reading his paper, I was led to consider deeply the definition of equality he then used, which runs as follows:

> two elements are said to be equal when either can replace the other in every proposition in which it occurs.

Though this definition seems very clear at one glance, yet it seems to me that it is not so on profound consideration. We shall try to explain this point in this paper. Moreover, in proving the theorems from his ten postulates, he used the propositions "if $A=B$ and $B=C$, then $A=B$" (1), "if $A=B$ and $C=D$, then $A+C=B+D$" (2), and others, without giving any mention of them, perhaps as natural results arising from the above definition of equality. But this is highly questionable; on the contrary, we may give a class of things satisfying the above ten postulates and his definition of equality, but not satisfying the above propositions (1), (2) and etc. (see Chapter III). For this and other reasons, I think it will be better to form a set of fundamental postulates which contains not only "addition and multiplication" as undefined operations, but also "equality" as an undefined relation, for a basis of constructing new systems of numbers by axiomatic method. I tried to do this and obtained *a fundamental set of postulates containing the above three undefined terms and having the three fundamental properties required in a set of postulates, namely independence, consistency, and sufficiency of postulates.* These are discussed in Chapter II.

In "The Fundamental Laws of Addition and Multiplication in Elementary Algebra," after giving the ten postulates concerning operations ($A1$, $A2$, $A3$, $A4$, $A5$; $M1$, $M2$, $M3$, $M4$, $M5$) and the six postulates concerning existence ($E1$, $E2$, $E3$, $E4$, $E5$, $E6$), Huntington remarks that the following postulate holds in all the types of algebra which he considered in his paper:

> Postulate $E7$. If $x \neq y$, then there is either an element v, such that $x=y+v$, or an element w such that $y=x+w$;

and also he remarks that, if this postulate be added to the list of the

above ten postulates, then $A\,4$ and $M\,3$ may be replaced by the following simpler postulates $A\,4'$ and $M\,3'$:

Postulate $A\,4'$. If $a \neq 0$, then $u\,a \neq 0$, where $u\,a$ is any multiple of a.

Postulate $M\,3'$. If $a \neq 0$, and $b \neq 0$, then $ab \neq 0$. ($x=0$ is an element satisfying the relation $x+x=x$).

After giving these remarks, he studied the relation of Postulates $M\,3$ and $M\,3'$, and constructed a system of numbers which satisfies Postulates $A\,1$, $A\,2$, $A\,3$, $A\,4$, $A\,5$, and $M\,1$, $M\,2$, $M\,3'$, $M\,4$, $M\,5$, but not $M\,3$; and thus he showed that the above set of postulates is "weakened" when $M\,3$ is replaced by $M\,3'$, since $M\,3$ cannot be deduced from $M\,3'$ and the other nine postulates without the aid of an additional postulate, like $E\,7$. As to the relation of the other postulates $A\,4$ and $A\,4'$, he made the following remark:

"an example of a system satisfying $A\,1$, $A\,2$, $A\,3$, $A\,4'$, $A\,5$, and $M\,1$, $M\,2$, $M\,3$, $M\,4$, $M\,5$, but not $A\,4$, would also be interesting. I have not, however, been able to find an example of this kind. It therefore remains an open question whether the above set of postulates is really weakened when $A\,4$ is replaced by $A\,4'$."

Now it seems to me that we can give such examples of the systems of things as Huntington wished to find. These will be discussed in Chapter III.

In Chapter IV, we shall give some singular systems of numbers, and also some properties of distributive law, considered from the axiomatic point of view.

CHAPTER I.

Systematic Developement of Number-Systems.

(A). Natural Numbers.

Introduction of Two Fundamental Numbers.

We start from *the existence of one fundamental thing and two fundamental operations combining this fundamental thing to itself;* and denote this hing by A and the two operations by $\oplus$ and $\otimes$ (which may be called addition and multiplication). The *simplest* result arising from the comination of A to A by one of the two operations $\oplus$ and $\otimes$ (say $\otimes$) is that it is identical with the original thing itself; we denote this fact by

the symbol "$(A \otimes A) \equiv A$." Then it is *most natural* to think that the other result arising from the other operation $\oplus$ is different from the original thing itself, since the operation $\oplus$ is different from the operation $\otimes$; we denote this fact by the symbol "$(A \oplus A) \not\equiv A$," and the result $A \oplus A$ by B. Thus we are led to set up the following two postulates connecting one undefined thing A and two undefined operations $\oplus$ and $\otimes$.

$$\text{(I)} \qquad (A \otimes A) \equiv A.$$

$$\text{(II)} \qquad (A \oplus A) \not\equiv A.$$

The thing A or $A \otimes A$ is called the first number and is denoted by A or A_1; and the thing $A \oplus A$ is called the second number and is denoted by B or A_2. *These two different numbers are fundamental ones in our theory; all other new numbers are constructed from these two numbers* in the following manner.

Construction of a Number-System.

First take a pair of numbers A, B, obtained above, and with this, construct the third new thing (A, B), which may be denoted by C; next take a pair of A, C and construct the fourth new thing (A, C), which may be denoted by D; and so on. The aggregate of all these things (A, B), $(A, C), \ldots$, together with two fundamental numbers $A \otimes A$, $A \oplus A$, forms a class of things called natural numbers.

For the sake of unification of the form, we denote the two fundamental numbers $A \otimes A$ and $A \oplus A$ by the symbols $(A \cdot A)$ and (A, A) respectively. But, as to the other new things (A, B), (A, C), $\ldots$, no property of them is known untill we shall set up the postulates and definitions to treat them as numbers.

The fundamental conditions that a class of things may be treated as a class of numbers are that (i) they may be compared with each other, and that (ii) they may be combined by fundamental operations. For these purposes, we lay down the following postulates and definitions as to the operations of our numbers and the comparison of them.

First, for the three relations "equality," "greater than," and "less than," we give the following definitions.

Definition (A). *Two numbers (A, M) and (A, N) are said to be equal with each other, when and only when M is identical with N.* The symbol of equality is denoted by "$=$" as usual.

Definition (B). *A number (A, M) is said to be less than a number (A, N), and conversely the number (A, N) is said to be greater than the number (A, M), when and only when (A, M) is formed before the number*

(A, N) is formed, in the process of construction of our number-system. The symbols of "greater than" and "less than" are denoted by "$>$" and "$<$" respectively as usual.

Secondly, for the operation $\oplus$ of the first number and any other number, we give the following postulate.

Postulate (I_a). *$(A. A) \oplus (A, M)$ denotes the number $(A, A \oplus M)$, or in symbol, $(A. A) \oplus (A, M) \equiv (A, A \oplus M)$* (1).

From this postulate, we may prove that any number (A, M) may be denoted in the form $(A \oplus (A \oplus (A \oplus \ldots \oplus A)))$ (Theorem 1). For the operation $\oplus$ of these numbers, we give the following postulate.

Postulate (I_b). *$(A \oplus A \oplus \ldots \oplus A) \oplus (A \oplus A \oplus A \oplus \ldots \oplus A)$ denotes the number $(A \oplus A \oplus \ldots \oplus A \oplus A \oplus A \oplus A \oplus \ldots \oplus A)$, or in symbol, $(A \oplus A \oplus \ldots \oplus A) \oplus (A \oplus A \oplus A \oplus \ldots \oplus A) \equiv (A \oplus A \oplus \ldots \oplus A \oplus A \oplus A \oplus A \oplus \ldots \oplus A)$; in other words, bracket may be omitted in this operation.*

From these postulates (I_a), (I_b), we may deduce the following law of addition (Theorem 6).

$$(A, M) \oplus (A, N) \equiv (A, A \oplus M \oplus N),$$

or

$$(A, M) \oplus (A, N) \equiv (A \oplus A, M \oplus N).$$

Thirdly, for the operation $\otimes$ of our numbers, we give the following postulates.

Postulate (II_a). $(A. A) \otimes (A, M) \equiv (A, M) \equiv (A, M) \otimes (A. A)$.

Postulate (II_b). $(A, M) \otimes (A, N) \equiv \{A, (M \oplus N) \oplus (M \otimes N)\}$.

Fundamental Properties of Our Number-System.

From the above postulates and definitions, we may deduce all the propositions concerning natural numbers.

Theorem 1. $(A, B) \equiv A \oplus B, (A, C) \equiv A \oplus C, \ldots, (A, M) \equiv A \oplus M, \ldots$

Proof. Put A for M in Postulate (I_a), then we have

(1) $$(A. A) \oplus (A, A) \equiv (A, A \oplus A).$$

But, by our convention, we have

$$(A. A) \equiv A, \qquad (A, A) \equiv A \oplus A \equiv B.$$

Therefore, from (1), it follows that

(2) $$A \oplus B \equiv (A, B).$$

Next put B for M in Postulate (I_a), then we have

$$(A. A) \oplus (A, B) \equiv (A, A \oplus B),$$

whence we have at once

(3) $$A \oplus C \equiv (A, C).$$

(1) This postulate may be replaced by $(A. A) \oplus (A, M) \equiv (A, (A, M))$.

using the relation (2) and the convention $(A, B) \equiv C$.

Proceeding in this way, we have generally

$$(A, M) \equiv A \oplus M.$$

Theorem 2. $(A, M) \equiv (A \oplus A \oplus \ldots\ldots \oplus A)$.

Proof. From Theorem 1 and Postulate (I_b), we have the relations

$$(A, B) \equiv (A \oplus B) \equiv \{A \oplus (A \oplus A)\} \equiv (A \oplus A \oplus A) \equiv C;$$
$$(A, C) \equiv (A \oplus C) \equiv \{A \oplus (A \oplus A \oplus A)\} \equiv (A \oplus A \oplus A \oplus A) \equiv D;$$
$$(A, D) \equiv (A \oplus D) \equiv \{A \oplus (A \oplus A \oplus A \oplus A)\} \equiv (A \oplus A \oplus A \oplus A \oplus A) \equiv E;$$

and so on.

For the sake of convenience, we make the following conventions.

The number A is denoted by A_1; and the number $A \oplus A$ by A_2; and the number $A \oplus A \oplus A$ by A_3; and so on; in general, $A_m \oplus A$ is denoted by A_{m+1}. Here no knowledge of these arabic numerals 1, 2, 3,.. ..is presupposed beyond the rule by which, when any one of them is given, the next following one can be written down, the rule being of such a nature that each new symbol is different from all that have gone before it. If p be any numeral mentioned above, the numeral next following p is called the successor of p and is denoted by $p+1$; the successor of $p+1$ is denoted by $p+2$; the successor of $p+2$ is denoted by $p+3$; and so on; in general, the successor of $p+q$ is denoted by $p+q+1$. Similarly, the numeral immediately before p is called the predecessor of p and is denoted by $p-1$; the predecessor of $p-1$ is denoted by $p-2$; and so on. When two arabic numerals m and n are identical with each other, they are said to be "equal" to each other and is denoted by "$m=n$." When arabic numerals are arranged in an order, if m is before n, then m is said to be "less than n" and is denoted by the symbol "$m<n$;" if m is after n, m is said to be "greater than n" and is denoted by the symbol "$m>n$."

From the above convention, it follows that

(1) $A_n \oplus A \equiv A_{n+1}$, $A_{n+1} \oplus A \equiv A_{n+2}, \ldots\ldots\ldots,$

(2) $(A, M) \equiv (A \oplus A \oplus A \oplus \ldots\ldots \oplus A) \equiv A_m$.

Theorem 3. $A_p \oplus A_q \equiv A_{p+q}$.

Proof.
$$\begin{aligned} A_p \oplus A_q &\equiv A_p \oplus (A \oplus A \oplus \ldots\ldots \oplus A) && \text{(by convention),}\\ &\equiv A_p \oplus A \oplus A \oplus \ldots\ldots \oplus A && \text{(by Postulate}(I_b)),\\ &\equiv A_{p+1} \oplus A \oplus \ldots\ldots \oplus A && \text{(by convention),}\\ &\equiv A_{p+2} \oplus A \ldots\ldots \oplus A && \text{(by convention),}\\ &\equiv \ldots\ldots\ldots\ldots\ldots\ldots && \\ &\equiv A_{p+q-1} \oplus A && \text{(by convention),}\\ &\equiv A_{p+q} && \text{(by convention),} \end{aligned}$$

Theorem 4. The associative law concerning the operation $\oplus$ holds good in our system.

$$\{(A, L)\oplus(A, M)\}\oplus(A, N)\equiv(A, L)\oplus\{(A, M)\oplus(A, N)\},$$

or

$$(A_p\oplus A_q)\oplus A_r\equiv A_p\oplus(A_q\oplus A_r).$$

Proof. $(A_p\oplus A_q)\oplus A_r\equiv\{(A\oplus A\oplus A\oplus A\oplus\ldots\oplus A)\oplus(A\oplus A\oplus\ldots\oplus A)\}\oplus(A\oplus\ldots\oplus A)$ (by convention),

$\equiv(A\oplus A\oplus A\oplus A\oplus\ldots\oplus A\oplus A\oplus A\oplus\ldots\oplus A)\oplus(A\oplus\ldots\oplus A)$ (by Postulate (I_b)),

(a) $\equiv(A\oplus A\oplus A\oplus A\oplus\ldots\oplus A\oplus A\oplus A\oplus\ldots\oplus A\oplus A\oplus\ldots\oplus A)$ (by Postulate (I_b)),

$A_p\oplus(A_q\oplus A_r)\equiv(A\oplus A\oplus A\oplus A\oplus\ldots\oplus A)\oplus\{(A\oplus A\oplus\ldots\oplus A)\oplus(A\oplus\ldots\oplus A)\}$ (by convention),

$\equiv(A\oplus A\oplus A\oplus A\oplus\ldots\oplus A)\oplus(A\oplus A\oplus\ldots\oplus A\oplus A\oplus\ldots\oplus A)$ (by Postulate (I_b)),

(b) $\equiv(A\oplus A\oplus A\oplus A\oplus\ldots\oplus A\oplus A\oplus A\oplus\ldots A\oplus A\oplus\ldots\oplus A\oplus\ldots\oplus A)$ (by Postulate (I_b)).

Now, that (a) and (b) are identical with each other is seen at once by making the first A of (a) correspond to the first A of (b) and the second A of (a) to the second A of (b) and so on(1). Therefore we have

$$(A_p\oplus A_q)\oplus A_r\equiv A_p\oplus(A_q\oplus A_r).$$

Cor. $$(p+q)+r=p+(q+r).$$

Proof. By Theorem 3, we have

$$(A_p\oplus A_q)\oplus A_r\equiv A_{p+q}\oplus A_r\equiv A_{(p+q)+r},$$
$$A_p\oplus(A_q\oplus A_r)\equiv A_p\oplus A_{q+r}\equiv A_{p+(q+r)},$$

and by Theorem 4, we have

$$(A_p\oplus A_q)\oplus A_r\equiv A_p\oplus(A_q\oplus A_r).$$

Therefore, it follows that

$$A_{(p+q)+r}\equiv A_{p+(q+r)},$$

whence again it follows that

$$(p+q)+r=p+(q+r).$$

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(1) For example, if we put $A_2\equiv A'\oplus A'$, $A_3\equiv A''\oplus A''\oplus A''$, $A_4\equiv A'''\oplus A'''\oplus A'''\oplus A'''$ for the sake of distinction, then (a) and (b) become

$$(a)\equiv(A'\oplus A'\oplus A''\oplus A''\oplus A''\oplus A'''\oplus A'''\oplus A'''\oplus A'''),$$
$$(b)\equiv(A'\oplus A'\oplus A''\oplus A''\oplus A''\oplus A'''\oplus A'''\oplus A'''\oplus A'''),$$

from which it may be seen at once that they are identical with each other.

Thus we have proved that both our number-system and the arabic numerals obey the associative law at the same time.

Theorem 5. The commutative law concerning the operation $\oplus$ holds good in our system.

$$(A, M) \oplus (A, N) \equiv (A, N) \oplus (A, M),$$

or

$$A_m \oplus A_n \equiv A_n \oplus A_m.$$

Proof. In the first place, we prove the following proposition

(1) $$A \oplus A_n \equiv A_n \oplus A$$

as a special case of our law.

Suppose that the proposition is true for $n=k$, and let us prove that the proposition is also true for $n=k+1$.

$$\begin{aligned} A \oplus A_{k+1} &\equiv A \oplus (A_k \oplus A) && \text{(by convention)},\\ &\equiv (A \oplus A_k) \oplus A && \text{(by Theorem 4)},\\ &\equiv (A_k \oplus A) \oplus A && \text{(by hypothesis)},\\ &\equiv A_{k+1} \oplus A && \text{(by Theorem 3)}. \end{aligned}$$

$$\therefore \quad A \oplus A_{k+1} \equiv A_{k+1} \oplus A.$$

Now, for $n=1$, the above proposition is obviously true; therefore by mathematical induction, the proposition is generally true.

Using the above proposition as lemma, we shall prove the general case of this law.

(2) $$A_m \oplus A_n \equiv A_n \oplus A_m.$$

Suppose that the proposition is true for $m=m'$, and let us prove that the proposition is also true for $m=m'+1$.

$$\begin{aligned} A_{m'+1} \oplus A_n &\equiv (A_{m'} \oplus A) \oplus A_n && \text{(by convention)},\\ &\equiv A_{m'} \oplus (A \oplus A_n) && \text{(by Theorem 4)},\\ &\equiv A_{m'} \oplus (A_n \oplus A) && \text{(by lemma)},\\ &\equiv (A_{m'} \oplus A_n) \oplus A && \text{(by Theorem 4)},\\ &\equiv (A_n \oplus A_{m'}) \oplus A && \text{(by hypothesis)},\\ &\equiv A_n \oplus (A_{m'} \oplus A) && \text{(by Theorem 4)},\\ &\equiv A_n \oplus A_{m'+1} && \text{(by Theorem 3)}; \end{aligned}$$

$$\therefore \quad A_{m'+1} \oplus A_n \equiv A_n \oplus A_{m'+1}.$$

But, by (1), for $m \equiv 1$, the proposition is true; therefore, by mathematical induction, it is generally true.

By the remark of Theorem 2, since (A, M) and (A, N) may be denoted by A_m and A_n respectively, we have the relation

$$(A, M) \oplus (A, N) \equiv (A, N) \oplus (A, M).$$

Cor. $m+n=n+m.$

Proof. $A_n \oplus A_m \equiv A_{n+m}$ (by Theorem 3),

$A_m \oplus A_n \equiv A_{m+n}$ (by Theorem 3).

But $A_n \oplus A_m \equiv A_m \oplus A_n$ (by Theorem 5).

Therefore $A_{n+m} \equiv A_{m+n}$,

accordingly we have the equality $n+m=m+n$.

Thus we see that both our number-system and arabic numerals obey the commutative law.

Cor. 2. If, in the number (A, M), M denotes A_n (theorem 2), then

$$(A, M) \equiv (A, A_m) \equiv A \oplus A_m \equiv A_m \oplus A \equiv A_{m+1}.$$

Theorem 6. General law of addition may be written as follows.

$$(A, M) \oplus (A, N) \equiv (A, A \oplus M \oplus N) \equiv (A \oplus A, M \oplus N),$$

or $$(A, A_m) \oplus (A, A_n) \equiv (A, A \oplus A_m \oplus A_n) \equiv (A \oplus A, A_m \oplus A_n).$$

Proof. $(A, A_m) \oplus (A, A_n) \equiv (A \oplus A_m) \oplus (A \oplus A_n)$ (by Theorem 1),

$\equiv A \oplus A_m \oplus A \oplus A_n$ (by Theorem 4),

$\equiv A \oplus (A_m \oplus A) \oplus A_n$ (by Theorem 4),

(a) $\equiv A \oplus (A \oplus A_m) \oplus A_n$ (by Theorem 5),

$\equiv A \oplus \{(A \oplus A_m) \oplus A_n\}$ (by Theorem 4),

$\equiv (A, A \oplus A_m \oplus A_n)$ (by Theorem 1).

Moreover, we have

$(A, A_m) \oplus (A, A_n) \equiv A \oplus (A \oplus A_m) \oplus A_n$ (by (a)),

$\equiv A \oplus A \oplus A_m \oplus A_n$ (by Theorem 4),

$\equiv (A \oplus A) \oplus (A_m \oplus A_n)$ (by Theorem 4),

$\equiv (A \oplus A, A_m \oplus A_n)$ (by Theorem 1).

Theorem 7. If (A, M) and (A, N) are any two elements of our system, then $\{(A, M) \oplus (A, N)\}$ is also an element of our system.

Proof. Put $(A, M) \equiv (A, A_m)$ and $(A, N) \equiv (A, A_n)$ (Theorem 2), then by Theorems 3 and 6, we have

$$(A, M) \oplus (A, N) \equiv (A, A_m) \oplus (A, A_n)$$

$$\equiv (A, A \oplus A_m \oplus A_n) \quad \text{(by Theorem 6)}$$

$$\equiv (A, A_{1+m+n}) \quad \text{(by Theorem 3).}$$

But A_{1+m+n} is an element of our system (Theorem 2); if we denote it by L, then (A, A_{1+m+n}) is denoted by (A, L), and so it is an element of our system; therefore $\{(A, M) \oplus (A, N)\}$ is also an element of our system.

From the above established theorems of our number-system and arabic numerals, we may prove the following two fundamental theorems concerning the operation $\oplus$.

Theorem 8. *If* $(A, A_m) \oplus (A, A_p) \equiv (A, A_m) \oplus (A, A_q)$, then $(A, A_p) \equiv (A, A_q)$.

Proof. To prove this theorem, it is sufficient to prove that $\{(A, A_m) \oplus (A, A_p)\}$ is greater than or less than $\{(A, A_m) \oplus (A, A_q)\}$ according as (A, A_p) is greater than or less than (A, A_q); for, when the above assertion is true, if (A, A_p) would not be identical with (A, A_q), then A_p would not be identical with A_q and so (A, A_p) would be greater than or less than (A, A_q) by the definitions of equality and inequality, and accordingly $\{(A, A_m) \oplus (A, A_p)\}$ would be greater than or less than $\{(A, A_m) \oplus (A, A_q)\}$, contrary to the hypothesis. Now, that the above assertion is true may be proved by mathematical induction as follows.

Suppose that the above assertion is true for a certain value m' of m, namely suppose that $\{(A, A_{m'}) \oplus (A, A_p)\}$ is greater than (or less than) $\{(A, A_{m'}) \oplus (A, A_q)\}$ according as (A, A_p) is greater than (or less than) (A, A_q), then we may prove that the same assertion is also true for the value of $m = m' + 1$. For, by Theorem 5, we have

$$(A, A_{m'}) \oplus (A, A_p) \equiv (A, A_p) \oplus (A, A_{m'}),$$
$$(A, A_{m'}) \oplus (A, A_q) \equiv (A, A_q) \oplus (A, A_{m'}).$$

Therefore, from the hypothesis, it follows that $\{(A, A_p) \oplus (A, A_{m'})\}$ is formed after (or before) $\{(A, A_q) \oplus (A, A_{m'})\}$ is formed; and so $\{(A, A_p) \oplus (A, A_{m'}) \oplus A\}$ must also be formed after (or before) $\{(A, A_q) \oplus (A, A_{m'}) \oplus A\}$ is formed. But we have

$$\begin{aligned}
(A, A_p) \oplus (A, A_{m'}) \oplus A &\equiv (A, A_p) \oplus \{(A, A_{m'}) \oplus A\} && \text{(by Theorem 4)},\\
&\equiv (A, A_p) \oplus \{(A \oplus A_{m'}) \oplus A\} && \text{(by Theorem 1)},\\
&\equiv (A, A_p) \oplus \{A \oplus (A_{m'} \oplus A)\} && \text{(by Theorem 4)},\\
&\equiv (A, A_p) \oplus (A \oplus A_{m'+1}) && \text{(by Theorem 3)},\\
&\equiv (A, A_p) \oplus (A, A_{m'+1}) && \text{(by Theorem 1)},\\
&\equiv (A, A_{m'+1}) \oplus (A, A_p) && \text{(by Theorem 5)}.
\end{aligned}$$

Similarly we have

$$(A, A_q) \oplus (A, A_{m'}) \oplus A \equiv (A, A_{m'+1}) \oplus (A, A_q).$$

Therefore, by the definition of inequality, $\{(A, A_{m'+1}) \oplus (A, A_p)\}$ is greater than (or less than) $\{(A, A_{m'+1}) \oplus (A, A_q)\}$. Thus the above assertion is true for $m = m' + 1$, when it holds good for $m = m'$.

Now the above assertion is obviously true for $m = 1$, therefore, by mathematical induction, it is generally true. Accordingly, Theorem 8 must also be true.

Cor. The same theorem is true for arabic numerals, namely if $k + r = k + s$, then $r = s$.

By Theorem 2, (A, A_m), (A, A_p), (A, A_q) may be denoted by A_k, A_r, A_s respectively; therefore Theorem 8 may be written in the form: "if $A_k \oplus A_r \equiv A_k \oplus A_s$, then $A_r \equiv A_s$." Again, by Theorem 3, it may be denoted in the form: "if $A_{k+r} \equiv A_{k+s}$, then $A_r \equiv A_s$"; whence follows at once the validity of our corollary.

Theorem 9. *If* $m(A, A_p) \equiv m(A, A_q)$, then $(A, A_p) \equiv (A, A_q)$. (Here $m(A, A_p)$ denotes the number $(A, A_p) \oplus (A, A_p) \oplus \ldots \oplus (A, A_p)$ repeated m times).

Proof. As in the proof of the above theorem, to prove this theorem, it is sufficient to prove that $m(A, A_p)$ is greater than (or less than) $m(A, A_q)$ according as (A, A_p) is greater than (or less than) (A, A_q). This assertion may be proved by mathematical induction as follows.

Suppose that $(m-1)(A, A_p)$ is greater than (or less than) $(m-1)(A, A_q)$ when (A, A_p) is greater than (or less than) (A, A_q), then by the definition of inequality, $(m-1)(A, A_p) \oplus A_{q+1}$ must be formed after (or before) $(m-1)(A, A_q) + A_{q+1} \equiv (m-1)(A, A_q) \oplus (A, A_q) \equiv m(A, A_q)$ is formed. Moreover, since $(A, A_p) \equiv A_{p+1}$ is greater than $(A, A_q) \equiv A_{q+1}$, by supposition, $(m-1)(A, A_p) \oplus A_{p+1} \equiv m(A, A_p)$ must be formed after (or before) $(m-1)(A, A_p) \oplus A_{q+1}$ is formed. Therefore, $m(A, A_p)$ must be formed after (or before) $m(A, A_q)$ is formed, so that, by the definition of inequality, $m(A, A_p)$ is greater than (or less than) $m(A, A_q)$.

Thus when our assertion is true for certain value of m, it is also true for the next following value of m. But our assertion is obviously true for $m=1$; therefore, by mathematical induction, it is generally true. Accordingly, Theorem 9 must also be true.

Cor. The same theorem is true for arabic numerals.

Thus far, we have proved the essential properties of our number-system and all fundamental theorems concerning the operation $\oplus$. Now we proceed to prove the fundamental theorems concerning another operation $\otimes$.

Theorem 10. If (A, M) and (A, N) are any two elements of our number-system, then $(A, M) \otimes (A, N)$ is also an element of our system.

Proof. Denote M and N by A_m and A_n respectively, and let us see whether $(A, A_m) \otimes (A, A_n)$ may be expressed in the form (A, A_l) or not.

In the first place, we have

$$\text{(a)} \qquad (A. A) \otimes (A, A_n) \equiv (A, A_n)$$

by Postulate (II_a). This relation may be written in the form

$$(A. A) \otimes (A, A_n) \equiv A_{n+1}.$$

by the law of addition.

Secondly, putting A for M and A_n for N in Postulate (II_b), we have

$$
\begin{aligned}
(A, A)\otimes(A, A_n) &\equiv \{A, (A\oplus A_n)\oplus(A\otimes A_n)\} & \\
&\equiv (A, A\oplus A_n\oplus A_n) & \text{(by (a))},\\
&\equiv (A, A_{1+n+n}). & \text{(by Theorem 3)},\\
&\equiv A\oplus A_{1+n+n} & \text{(by Theorem 1)},\\
&\equiv A_{1+n+n+1} & \text{(by Theorem 3)},\\
&\equiv A_{(n+1)+(n+1)} & \text{(by law of addition concerning numerals)}.
\end{aligned}
$$

Here we make a convention that $n+n$ is denoted by the symbol $n.2$, and $n+n+n$ by the symbol $n.3$, and, in general, $n+n+n\ldots.+n$ repeated m times by the symbol $n.m$. By using this notation, the last result may be written in the form

$$\text{(b)} \quad (A, A)\otimes(A, A_n)\equiv A_{(n+1)+(n+1)}\equiv A_{(n+1).2},$$

Thirdly, putting A_2 for M and A_n for N in Postulate (II_b), we have

$$
\begin{aligned}
(A, A_2)\otimes(A, A_n) &\equiv \{A, (A_2\oplus A_n)\oplus(A_2\otimes A_n)\}, & \\
&\equiv \{A, A_{2+n}\oplus(A_2\otimes A_n)\} & \text{(by Theorem 3)}.
\end{aligned}
$$

But by (b) we have

$$(A_2\otimes A_n)\equiv(A, A)\otimes(A, A_{n-1})\equiv A_{n.2}.$$

Therefore

$$
\text{(c)} \qquad
\begin{aligned}
(A, A_2)\otimes(A, A_n) &\equiv (A, A_{2+n}\oplus A_{n.2}), & \\
&\equiv (A, A_{2+n+n.2}) & \text{(by Theorem 3)},\\
&\equiv (A, A_{2+n.3}) & \text{(by convention)},\\
&\equiv A\oplus A_{2+n.3} & \text{(by Theorem 1)},\\
&\equiv A_{1+2+n.3} & \text{(by Theorem 3)},\\
&\equiv A_{(n+1)+(n+1)+(n+1)} & \text{(by law of addition)},\\
&\equiv A_{(n+1).3} & \text{(by convention)}.
\end{aligned}
$$

The results (a), (b), (c) may also be written as follows.

$$
\begin{aligned}
A_1\otimes A_{n+1}&\equiv A_{(n+1).1},\\
A_2\otimes A_{n+1}&\equiv A_{(n+1).2},\\
A_3\otimes A_{n+1}&\equiv A_{(n+1).3}.
\end{aligned}
$$

Similarly, we may prove that

$$
\begin{aligned}
A_{n+1}\otimes A_1&\equiv A_{(n+1).1},\\
A_{n+1}\otimes A_2&\equiv A_{(n+1).2},\\
A_{n+1}\otimes A_3&\equiv A_{(n+1).3}.
\end{aligned}
$$

Now we shall prove that the above law holds generally, namely the formula $A_m\otimes A_n\equiv A_{m+m+\ldots.+m}\equiv A_{m.n}$ holds generally.

Suppose that the law is true for $m \leqq m'$ and $n \leqq n',$ then we have

$$
\begin{aligned}
A_{m'+1} \otimes A_{n'+1} &\equiv (A, A_{m'}) \otimes (A, A_{n'}) && \text{(by Theorems 1 and 3),} \\
&\equiv \{A, (A_{m'} \oplus A_{n'}) \oplus (A_{m'} \otimes A_{n'})\} && \text{(by Postulate } II_b\text{),} \\
&\equiv \{A, A_{m'} \oplus A_{n'} \oplus A_{m'n'}\} && \text{(by hypothesis),} \\
&\equiv A \oplus (A_{m'} \oplus A_{n'} \oplus A_{m'n'}) && \text{(by Theorem 1),} \\
&\equiv A \oplus A_{m'} \oplus (A_{n'} \oplus A_{m'n'}) && \text{(by Theorem 4),} \\
&\equiv A \oplus A_{m'} \oplus \{(A \oplus A \oplus \ldots . n' \text{ times}) \oplus \\
&\qquad (A_{m'} \oplus A_{m'} \oplus \ldots . n' \text{ times})\} && \text{(by Theorem 2 and hypothesis),} \\
&\equiv (A \oplus A_{m'}) \oplus \{(A \oplus A_{m'}) \oplus (A \oplus A_{m'}) \\
&\qquad \oplus \ldots . n' \text{ times}\} && \text{(by Theorems 4 and 5)} \\
&\equiv A_{m'+1} \oplus \{A_{m'+1} \oplus A_{m,+1} \oplus \ldots . n' \text{ times}\} && \text{(by Theorem 3)} \\
&\equiv A_{m'+1} \oplus A_{m'+1} \oplus A_{m'+1} \oplus \ldots . n'+1 \text{ times} && \text{(by Theorem 4)} \\
&\equiv A_{(m'+1)+(m'+1)+\ldots\ldots n'+1 \text{ times}} && \text{(by Theorem 3)} \\
&\equiv A_{(m'+1)(n'+1)} && \text{(by convention).}
\end{aligned}
$$

Similarly, we can prove that

$$A_{m'+1} \otimes A_{n'} \equiv A_{(m'+1)n'} \text{ and } A_{m'} \otimes A_{n'+1} \equiv A_{m'(n'+1)}.$$

Therefore, if the above law holds good for certain values m' and n' of m and n and all values preceding them, then it also holds good for next following values of them. But the law is true for $m=1$ and any value of n; and also for any value of m and $n=1$; namely the relations

$$
\begin{aligned}
A_1 \otimes A_n \equiv A_n \equiv A_{1.n} \\
A_n \otimes A_1 \equiv A_n \equiv A_{n.1}
\end{aligned}
$$

hold good always by Postulate (II_a). Therefore, by mathematical induction, the law is generally true. This law is very important; we shall call it the law of multiplication.

By means of this law, we have the following relations.

$$
\begin{aligned}
(A, A_m) \otimes (A, A_n) &\equiv \{A, (A_m \oplus A_n) \oplus (A_m \otimes A_n)\}, \\
&\equiv (A, A_{m+n} \oplus A_{mn}), \\
&\equiv (A, A_{m+n+mn}).
\end{aligned}
$$

But A_{m+n+mn} is a number which is produced by operating the operation $\oplus$ to A, a certain number of times; and any number, produced by operating the operation $\oplus$ to A, any number of times, is a number of our number-system. Therefore A_{m+n+mn} is a number belonging to our

system, so that we may denote it by L. Thus we have the relation

$$(A, A_m) \otimes (A, A_n) \equiv (A, L).$$

Therefore $(A, A_m) \otimes (A, A_n)$ is also an element of our number-system.

Theorem 11. The commutative law concerning the operation $\otimes$ holds good in our number-system.

Proof. Since (A, M) and (A, N) may be denoted by A_m and A_n (Theorem 2), the commutative law $(A, M) \otimes (A, N) \equiv (A, N) \otimes (A, M)$ may be written in the form $A_m \otimes A_n \equiv A_n \otimes A_m$. We shall prove this law in the latter form.

Suppose that the above law is true for $n = n'$, then we have

$$\begin{aligned}
(1) \quad A_m \otimes A_{n'+1} &\equiv A_{m(n'+1)} && \text{(by law of multiplication),}\\
&\equiv A_{m+m+\ldots\ldots n'+1 \text{ times}} && \text{(by convention),}\\
&\equiv A_{(m+m+\ldots\ldots n' \text{ times})+m} && \\
&\equiv A_{mn'+m} && \text{(by convention),}\\
&\equiv A_{mn'} \oplus A_m. && \text{(by Theorem 3).}
\end{aligned}$$

$$\begin{aligned}
(2) \quad A_{n'+1} \otimes A_m &\equiv A_{(n'+1)+(n'+1)+\ldots\ldots m \text{ times}},\\
&\equiv A_{(n'+n'+\ldots\ldots m \text{ times})+(1+1+\ldots\ldots m \text{ times})},\\
&\equiv A_{n'm+m},\\
&\equiv A_{n'm} \oplus A_m.
\end{aligned}$$

But, by hypothesis, we have

$$A_m \otimes A_{n'} \equiv A_{n'} \otimes A_m,$$

or

$$A_{mn'} \equiv A_{n'm}.$$

Therefore, from (1) and (2), we have at once

$$A_m \otimes A_{n'+1} \equiv A_{n'+1} \otimes A_m.$$

Thus, if the law holds good for a certain value of n, then it also holds good for the next following value of n. But, for $n=1$, the above law is true by Postulate (II_a); therefore, by induction, the law is generally true.

Cor. $$m.\,n = n.m.$$

Theorem 12. The distributive law holds good in our number-system.

Proof. We have to prove the relation

$$(A, L) \otimes \{(A, M) \oplus (A, N)\} \equiv \{(A, L) \otimes (A, M)\} \oplus \{(A, L) \otimes (A, N)\}.,$$

or the relation $$A_l \otimes (A_m \oplus A_n) \equiv (A_l \otimes A_m) \oplus (A_l \otimes A_n).$$

In the first place, we have the relation

$$A_l \otimes (A_n \oplus A_n) \equiv A_l \otimes A_{m+n} \qquad \text{(by law of addition)},$$
$$\equiv A_{m+n} \otimes A_l \qquad \text{(by commutative law)},$$
$$\equiv A_{(m+n)+(m+n)+\ldots\ldots l \text{ times}} \qquad \text{(by law of multiplication)}.$$

But by repeated application of associative and commutative laws of arabic numerals, we have the relation

$$(m+n)+(m+n)+\ldots\ l \text{ times} \equiv m+n+m+n+\ldots\ldots,$$
$$\equiv (m+m+\ldots\ldots l \text{ times}) + (n+n+\ldots\ldots l \text{ times}),$$
$$\equiv ml+nl,$$
$$\equiv lm+ln.$$

Substituting this result in the above, we have

$$\text{(a)} \qquad A_l \otimes (A_m \oplus A_n) \equiv A_{lm+ln}.$$

But, on the other hand, we have the relation

$$\text{(b)} \quad (A_l \otimes A_m) \oplus (A_l \otimes A_n) \equiv A_{lm} \oplus A_{ln} \equiv A_{lm+ln}$$

by the laws of addition and multiplication. From (a) and (b) we have at once the required relation

$$A_l \otimes (A_m \oplus A_n) \equiv (A_l \otimes A_m) \oplus (A_l \otimes A_n).$$

Cor. 1. $\quad (A_m \oplus A_n) \otimes A_l \equiv (A_m \otimes A_l) \oplus (A_n \otimes A_l).$

Cor. 2. $\quad l(m+n) = lm + ln.$

$$(m+n)l = ml + nl.$$

Theorem 13. The associative law concerning the operation $\otimes$ holds good in our number-system.

Proof. We have to prove the relation

$$\{(A, L) \otimes (A, M)\} \otimes (A, N) \equiv (A, L) \otimes \{(A, M \otimes (A, N)\},$$

or the relation $\qquad (A_l \otimes A_m) \otimes A_n \equiv A_l \otimes (A_m \otimes A_n).$

This may be proved by mathematical induction as follows.

First we shall prove this law for any values of l and a special value 1 of n. In this case, we have

$$(A_l \otimes A_m) \otimes A \equiv A_{lm} \otimes A \qquad \text{(by law of multiplication)},$$
$$\equiv A_{lm} \qquad \text{(by Postulate } (II_a)),$$
$$A_l \otimes (A_m \otimes A) \equiv A_l \otimes A_m \qquad \text{(by Postulate } (II_a)),$$
$$\equiv A_{lm} \qquad \text{(by law of multiplication)}.$$

$$\therefore \quad (A_l \otimes A_m) \otimes A \equiv A_l \otimes (A_m \otimes A). \qquad \text{(a)}$$

By using this result, we shall prove this law for any values of l, m,

and n. Suppose that this law is true for a certain value n' of n and for any values of l and m, then we have

$$(A_l \otimes A_m) \otimes A_{n'} \equiv A_l \otimes (A_m \otimes A_{n'})$$

and

$$A_{(lm)n'} \equiv A_{l(mn')}$$

by the hypothesis and law of multiplication. Moreover, we have the following relations.

$$\begin{aligned}
(A_l \otimes A_m) \otimes A_{n'+1} &\equiv A_{lm} \otimes A_{n'+1} && \text{(by law of multiplication)},\\
&\equiv A_{lm+lm+\ldots\ldots n'+1\ times} && \text{(by law of multiplication)},\\
&\equiv A_{(lm+lm+\ldots\ldots n'times)+lm} && \text{(by convention)},\\
&\equiv A_{(lm)\,n'+lm} && \text{(by convention)},\\
&\equiv A_{(lm)n'} \oplus A_{lm} && \text{(by Theorem 3)};\\
A_l \otimes (A_m \otimes A_{n'+1}) &\equiv A_l \otimes A_{mn'+m} && \text{(by the same reason as above)},\\
&\equiv A_{l(mn'+m)} && \text{(by law of multiplication)},\\
&\equiv A_{l\,(mn')+lm} && \text{(by distributive law of arabic numerals)},\\
&\equiv A_{l\,(mn')} \oplus A_{lm} && \text{(by Theorem 3)},\\
&\equiv A_{(lm)n'} \oplus A_{lm} && \text{(by hypothesis)}.
\end{aligned}$$

Therefore, we have

$$(A_l \otimes A_m) \otimes A_{n'+1} \equiv A_l \otimes (A_m \otimes A_{n'+1}).$$

But when $n=1$, the law is true by (a) for any values of l and m; therefore, by mathematical induction, it is generally true for any values of l, m and n.

Cor. $$(lm)n \equiv l(mn).$$

Theorem 14. If $(A, M) \otimes (A, P) \equiv (A, M) \otimes (A, Q)$, then $(A, P) \equiv (A, Q)$; *or if* $A_m \otimes A_p \equiv A_m \otimes A_q$, *then* $A_p \equiv A_q$.

Proof. To prove this theorem, it is sufficient to prove that $A_m \otimes A_p$ is greater than or less than $A_m \otimes A_q$ according as A_p is greater than or less than A_q, since any two elements of our system satisfies one and only one relation of "equality," "greater than," and "less than." By the property of our number-system, when A_p is greater than A_q, we have the relation $A_p \equiv A_q \oplus A_r$; and when A_p is less than A_q, we have the relation $A_q \equiv A_p \oplus A_r$.

We shall first consider the case where A_p is greater than A_q. In this case we have

$$A_m \otimes A_p \equiv A_m \otimes (A_q \oplus A_r),$$
$$\equiv (A_m \otimes A_q) \oplus (A_m \otimes A_r) \qquad \text{(by distributive law)},$$
$$\equiv (A_m \otimes A_q) \oplus A_{mr} \qquad \text{(by law of multiplication)}.$$

Therefore $A_m \otimes A_p$ is greater than $A_n \otimes A_q$.

The case in which A_p is less than A_q may similarly be treated. Thus the theorem is proved.

Cor. If $m \times p = m \times q$, then $p = q$.

Theorem 15. If $(A, M) \equiv (A, N)$ and $(A, P) \equiv (A, Q)$, *then*

$$(A, M) \oplus (A, P) \equiv (A, N) \oplus (A, Q),$$
$$(A, M) \otimes (A, P) \equiv (A, N) \otimes (A, Q).$$

This theorem may easily be proved by using the laws of addition and multiplication of our number-system.

Hitherto we have proved all the fundamental theorems concerning the operations $\oplus$ and $\otimes$. Now we proceed to consider the fundamental relations concerning equality and inequality. In my paper(1) called "Set of Independent Postulates concerning Equality and Inequality" I have proved that all the propositions concerning equality and inequality may be deduced from the set of the following five independent postulates.

I. Any two elements of the class satisfy at least one of the three relations "equal to," "greater than," and "less than."

II. Any element of the class has at least one element which is equal to it.

III. If A is less than B, then B is greater than A.

IV. If A is equal to B and B is greater than C, then A is greater than C.

V. If A is greater than B and B is greater than C, then A is greater than C.

By the construction of our number-system and definitions of equality and inequality, we can easily see that the above five propositions hold good in our number-system. Therefore all propositions concerning equality and inequality hold also in our system.

Theorems 4, 5, 7, 8, 9, 10, 11, 12, 13, 14 are ten fundamental laws concerning addition and multiplication given by Huntington, from which he deduced all other propositions concerning the above operations. By these theorems and the above five fundamental propositions of equality

(1) The Tôhoku Mathematical Journal, Vol. 14, Nos. 3,4, 1918.

and inequality, all propositions concerning the operations and comparison of natural numbers may be deduced in the usual manner. Therefore we shall not enter into further deduction.

In the course of our discussion, we have introduced arabic numerals having one-to-one correspondence to the elements of our number-system, and by the way, we have proved that the system of arabic numerals also obeys all fundamental laws of addition, multiplication, equality and inequality, when $(+)$, $(\times)$, $(=)$, $(\gtreqless)$ are defined as in our discussion. Therefore, we may take this system of arabic numerals as representative of the system of natural numbers.

We have constructed our number by taking *in pair the first number A and a number M already constructed.* Now we ask what number is obtained, if *any two numbers of our constructed class of natural numbers are taken in pair.* The answer is as follows.

When we take any two elements M, N of our natural numbers and form a pair of these numbers (M, N), if we postulate that the symbol (M, N) expresses the same kind of combination of M and N as that of A and M in our natural number (A, M), then the pair $(M. N)$ is an element of our natural numbers also, and the laws of their operations may be written in the form

$$(M, N)\oplus(P, Q)\equiv(M\oplus P, N\oplus Q),$$
$$(M, N)\otimes(P, Q)\equiv\{(M\otimes P)\oplus(N\otimes Q), (M\otimes Q)\oplus(N\otimes P)\};$$

or if we take arabic numerals m, n, p, q, then they may be written as follows.

$$(m, n)\oplus(p, q)\equiv(m+p, n+q)$$
$$(m, n)\otimes(p, q)\equiv(mp+nq,\ mq+np).$$

For, by Theorem 1, (A, M) denotes the combination $A\oplus M$, so that (M, N) denotes a number $M\oplus N$, which is an element of our natural numbers by Theorem 7. Moreover, the above laws of operations may easily be proved by using the commutative, associative and distributive laws of addition and multiplication.

Inverse Operations.

As it is used in the introduction of integers and rational numbers in after-times, here we shall introduce the inverse operations of addition and multiplication, and add a few propositions concerning them which are needed in later discussion.

Subtraction. When (A, M) is greater than (A, N), by the definition

of inequality and by the property of our number, there exists a natural number X, such that

$$(A, M) \equiv (A, N) \oplus X.$$

The operation of finding X when (A, M) and (A, N) are given is called subtraction and is denoted by the symbol $\ominus$, and the required number X is denoted by the symbol $(A, M) \ominus (A, N)$. This operation is an inverse operation of addition. That this operation $(A, M) \ominus (A, N)$ is impossible when (A, M) is equal to (A, N) or is less than (A, N) may be seen at once from the definitions of equality and inequality. With regard to this operation, we prove the following theorem, since it will be used in later discussion.

Theorem 16. If (A, M) is greater than (A, M') and moreover $(A, M) \oplus (A, N)$ is equal to $(A, M') \oplus (A, N')$, then $(A, M) \ominus (A, M')$ is equal to $(A, N') \ominus (A, N)$.

Proof. Since (A, M) is greater than (A, M') by hypothesis, $(A, M) \oplus (A, N)$ is greater than $(A, M') \oplus (A, N)$. Accordingly (A, N') must be greater than (A, N), in order that $(A, M) \oplus (A, N)$ may be equal to $(A, M') \oplus (A, N')$. Thus there exist two natural numbers X, Y, such that they satisfy the relations

$$(A, M) \equiv (A, M') \oplus X, \quad (A, N') \equiv (A, N) \oplus Y.$$

Substitute these relations in the equality

$$(A, M) \oplus (A, N) = (A, M') \oplus (A, N'),$$

then we have

$$(A, M') \oplus X \oplus (A, N') = (A, M') \oplus \{(A, N) \oplus Y\}.$$

But $\quad (A, M') \oplus X \oplus (A, N) \equiv (A, M') \oplus (A, N) \oplus X \quad$ (by commutative law),

$(A, M') \oplus \{(A, N) \oplus Y\} \equiv (A, M') \oplus (A, N) \oplus Y \quad$ (by associative law).

$$\therefore \quad \{(A, M') \oplus (A, N)\} \oplus X \equiv \{(A, M') \oplus (A, N)\} \oplus Y.$$

$$\therefore \quad X = Y \qquad \text{(by Theorem 8)}.$$

$$\therefore \quad (A, M) \ominus (A, M') = (A, N') \ominus (A, N).$$

Division. When the relation $(A, M) \otimes (A, N) \equiv (A, P)$ holds good, (A, P) is called a multiple of (A, M) or of (A, N). From this definition, it follows at once that when (A, P) is a multiple of (A, M), there exists a natural number X, such that the relation $(A, M) \otimes X \equiv (A, P)$ holds good. The operation of finding X when (A, M) and (A, P) are given is called division; it is an inverse operation of multiplication. This operation is

denoted by the symbol $\oslash$, and X is denoted by the symbol $(A, P) \oslash (A, M)$.

Now when two numbers (A, P) and (A, M) are given, the case may occur, such that the relation $(A, M) \otimes X \equiv (A, P)$ never holds good, whatever natural number is substituted for X. For example, take A_2 as (A, M) and A_5 as (A, P), then there is no natural number X satisfying the relation

$$A_2 \otimes X \equiv A_5.$$

Thus we have arrived at the following well known result.

Theorem 17. In the system of natural numbers, two direct operations (addition and multiplication) can be performed without any restriction. But their inverse operations (subtraction and division) can be performed in only limited cases.

Existence of Particular Elements.

Definition 1. If a number A satisfies the relation $A \oplus A \equiv A$, then A is called the zero-element (or zero) of the number-system considered.

Definition 2. If a number A, which is not a zero-element, satisfies the relation $A \otimes A \equiv A$, then A is called the unit-element of the number-system considered.

Theorem 18. In the system of natural numbers, there exists one and only one unit-element, but none of zero-element.

Proof. By our postulate, the element $A \equiv (A. A)$ satisfies the relation $A \otimes A \equiv A$; therefore this element is a unit-element of our system. But there is no other element satisfying the above relation. For, if there were such another element, denote it by B, then we have

$$B \otimes B \equiv B,$$

and accordingly we have

$$B \equiv B \otimes B$$

by the proposition of equality (if $P = Q$, then $Q = P$). Now, from the two relations

$$A \otimes A \equiv A \qquad B \equiv B \otimes B,$$

we have the relation

$$(A \otimes A) \otimes B \equiv A \otimes (B \otimes B)$$

by Theorem 15;

$$\therefore \quad A \otimes (A \otimes B) \equiv (A \otimes B) \otimes B \qquad \text{(by associative law);}$$

$$\therefore \quad A \otimes (B \otimes A) \equiv (A \otimes B) \otimes B \qquad \text{(by commutative law);}$$

$$\therefore \quad (A \otimes B) \otimes A \equiv (A \otimes B) \otimes B \qquad \text{(by associative law);}$$

$$\therefore \quad A \equiv B \qquad \text{(by Theorem 14).}$$

Therefore there is only one unit-element.

Further, any number A_n of our system satisfies the relation $A_n \oplus A_n \equiv A_{2n}$; but A_{2n} is greater than A_n and so cannot be equal to A_n by our definition, so that there is no zero-element in our system.

Non-Contradiction of Our Number-System.

Now let us consider whether our system of numbers is free from contradiction. Since we have to discuss the non-contradiction of properties of natural numbers, we cannot use any property of natural numbers; so we must take recourse from admitted and fundamental properties of our mind, or else from certain concrete things admitted to exist and to have certain properties.

If we admit that our mind has the following three properties:

(i) it can conceive one thing (or oneness),

(ii) it can combine([1]) (add) oneness to oneness repeatedly,

(iii) it can correspond a class of things to another class of things, and also if we admit that logical reasoning of our mind contains no contradiction in itself, then we can mentally construct a class of things (abstract) from one thing (oneness) satisfying all our postulates of natural numbers.

Denote oneness by A and the combination of oneness to oneness by $A \circledcirc A$, then since this combination can be made repeatedly by (ii), we can construct a class of abstract things $\{A, A \circledcirc A, A \circledcirc A \circledcirc A, \ldots .\}$ by this method. Here we may set up the law of operations of any two elements of this class by means of correspondence.

From this class of things, take any two things, for example, $L \equiv (A \circledcirc A \circledcirc A)$ and $M \equiv (A \circledcirc A \circledcirc A \circledcirc A)$; and to the combination of these two things $(A \circledcirc A \circledcirc A) \circledcirc (A \circledcirc A \circledcirc A \circledcirc A)$, correspond a thing of this class $K \equiv A \circledcirc A \circledcirc \ldots \circledcirc A$, which has the following correspondence to $L \circledcirc M$. The first element of L corresponds to the first element of K, next element of L to the next element of K and so on; when the elements of L are all taken up, the first element of M corresponds to the next succeeding one of K and so on; and they are in such a correspondence that, when all elements of M are taken up, so also does elements of K. In this case, we say that the combination $L \circledcirc M$ of two things L and M produces the thing K. We shall call such a mode of combination the addition of this class of things.

([1]) This combination is of such nature that the result of the combining of oneness to itself is different from oneness.

From the first kind of combination $\circledcirc$, we derive the second kind of combination by means of correspondence again. First we take any two things $L \equiv (A \circledcirc A \circledcirc A)$ and $M \equiv (A \circledcirc A \circledcirc A \circledcirc A)$, and from these, we construct the third thing $N \equiv (L \circledcirc L \circledcirc L \circledcirc L)$, such that the first element L of N corresponds to the first element A of M, and the next element L of N to the next element A of M and so on, untill all elements L's of N are taken up when all elements A's of M are also taken up. This constructed thing $N \equiv (L \circledcirc L \circledcirc L \circledcirc L)$ may be writen in the form $N \equiv (A \circledcirc A \circledcirc A) \circledcirc (A \circledcirc A \circledcirc A) \circledcirc (A \circledcirc A \circledcirc A) \circledcirc (A \circledcirc A \circledcirc A)$ and may be corresponded to the thing $K \equiv A \circledcirc A \circledcirc A \circledcirc \ldots \circledcirc A$ by the correspondence established in the first mode of combination. This mode of producing the thing K from the given thing L and M is called the second mode of combination of L and M. This combination may be called multiplication and may be denoted by the symbol $\odot$.

As to the comparison of the elements of this class, we give the same definitions of equality and inequality of them as those of natural numbers.

Thus, by the fundamental properties of our mind, we are able to construct a mentally existed class of things, and to derive the possibility of their operations and comparison, and to secure the uniqueness of result of operations.

Now with the operations $\circledcirc$, $\odot$ and the relations $=, >, <$ above established, it is easy to see that the above constructed class of mental things satisfies all postulates and fundamental propositions of our natural numbers when the operation $\circledcirc$ is substituted for $\oplus$ and the operation $\odot$ for $\otimes$.

Therefore, if we admit the above properties of our mind, we can construct a class of things which implies no contradiction, and which satisfies all our postulates. Hence our set of postulates neither contains contradiction nor lead to contradiction; and there exists at least one class of mental things satisfying all our postulates and fundamental propositions.

Further, if we admit the existence of concrete things having certain properties, we may give several classes of things satisfying our requirement.

Independence of Our Postulates.

We have given the two postulates (I_a) (I_b) for the operation $\oplus$, and the two postulates (II_a) (II_b) for the operation $\otimes$. The postulates (I_a) and (II_a) define the operations between the first number and any other numbers, while the postulates (I_b) and (II_b) in general define the operations between any numbers other than the first number. Since the first

number has the peculiar property that $A \otimes A$ is identical with itself and so we have denoted it by the symbol $(A.A)$, different from those of the other numbers, we have thought that it is convenient to express the law of operations in separate formulæ. Therefore, the above postulates may be considered, in their substance, as one postulate for each operation, and accordingly we may expect that they will be independent of one another. That they are really so may be proved rigorously as follows.

In the preceding discussion, we have constructed a non-contradictory class of things called natural numbers and especially a class of numerals as a representative of it; and we have given the laws of operating and comparing them as numbers. Therefore, in our foregoing discussion, we may use this class of numbers; and from it, we may construct other systems of numbers obeying other laws of operations. With these pseudo-systems of numbers, we may prove the independence of our postulates as follows.

Theorem (*a*). *Postulate* (II_a) *is independent of Postulates* (I_a), (I_b) *and* (II_b).

Proof. We consider the class of arabic numerals 1, 2, 3, ... , and denote any two elements of it by m and n. The results of operating the fundamental operations (addition and multiplication) to these elements m and n are already defined, and they are at once known when m and n are given; we denote these results by $m+n$ and $m \times n$. Now, by means of these results, we introduce the new laws of operations as given below.

$$m \oplus n \equiv m+n,$$

$$m \otimes n \equiv m \times n + m - n\text{(}^{1}\text{)},$$

$$(1, m) \equiv 1 \oplus m.$$

Then the class of numerals with these new laws of operations satisfies Postulates (I_a), (I_b), (II_b), and Postulate (A) $(1 \oplus 1 \neq 1)$, but not Postulate (II_a).

1. It satisfies Postulate (I_a).

By the given law of operations, we have

$$(1.1) \oplus (1, m) \equiv 1 \oplus (1 \oplus m) \equiv 1 \oplus (1+m) \equiv 1+(1+m) \equiv 1+1+m,$$

and $\quad (1, 1 \oplus m) \equiv 1 \oplus (1 \oplus m) \equiv 1+1+m;$

$\therefore \quad (1.1) \oplus (1, m) \equiv (1, 1 \oplus m).$

---

(1) $-$ denotes the subtraction of numerals.

But this is Postulate (I_a).

2. It satisfies Postulate (I_b).

$$(1\oplus 1\oplus \ldots\ldots \oplus 1)\oplus(1\oplus 1\oplus \ldots\ldots \oplus 1)$$
$$\equiv(1+1+\ldots\ldots+1)\oplus(1+1+\ldots\ldots+1)$$
$$\equiv m\oplus n\equiv m+n,$$
$$(1\oplus 1\oplus \ldots\ldots \oplus 1\oplus 1\oplus 1\oplus \ldots\ldots \oplus 1)$$
$$\equiv(1+1+\ldots\ldots+1+1+1+\ldots\ldots+1)$$
$$\equiv m+n;$$
$$\therefore\quad (1\oplus 1\oplus \ldots\ldots \oplus 1)\oplus(1\oplus 1\oplus \ldots\ldots \oplus 1)$$
$$\equiv(1\oplus 1\oplus \ldots\ldots \oplus 1\oplus 1\oplus \ldots\ldots \oplus 1).$$

But this is Postulate (I_b).

3. It satisfies Postulate (II_b).

$$(1, m)\otimes(1, n)\equiv(1\oplus m)\otimes(1\oplus n)$$
$$\equiv(1+m)\otimes(1+n)\equiv(1+m)\times(1+n)$$
$$+(1+m)-(1+n)\equiv 1+2m+m\times n,$$
$$\{1, (m\oplus n)\oplus(m\otimes n)\}\equiv 1\oplus\{(m+n)\oplus(m\times n+m-n)\}$$
$$\equiv 1+m+n+m\times n+m-n\equiv 1+2m+m\times n;$$
$$\therefore\quad (1, m)\otimes(1, n)\equiv\{1, (m\oplus n)\oplus(m\otimes n)\}.$$

But this is Postulate (II_b).

4. It satisfies Postulate (A).

$$1\oplus 1\equiv 2\not\equiv 1.$$

5. It does not satisfy Postulate (II_a).

$$(1, m)\otimes(1.1)\equiv(1\oplus m)\otimes 1\equiv(1+m)\otimes 1$$
$$\equiv(1+m)\times 1+(1+m)-1\equiv 1+2m,$$
$$(1, m)\equiv 1\oplus m\equiv 1+m;$$
$$\therefore\quad (1, m)\otimes(1.1)\not\equiv(1, m).$$

Moreover, we have

$$(1.1)\otimes(1, m)\equiv 1\otimes(1\oplus m)\equiv 1\otimes(1+m)$$
$$\equiv 1+m+1-(1+m)\equiv 1;$$
$$\therefore\quad (1, m)\otimes(1.1)\not\equiv(1.1)\otimes(1, m)$$

and $\quad (1.1)\otimes(1, m)\not\equiv(1, m).$

Now Postulate (II_a) may be divided into the three propositions

$$\text{(i)}\quad (1, m) \otimes (1.1) \equiv (1.1) \otimes (1, m),$$
$$\text{(ii)}\quad (1.1) \otimes (1, m) \equiv (1, m),$$
$$\text{(iii)}\quad (1, m) \otimes (1.1) \equiv (1, m);$$

and the above demonstration shows that no one of these three propositions cannot be deduced from Postulates (I_a), (I_b), (II_b) and (A). But, of these three, each of the propositions (ii) and (iii) can be deduced from the other two. Therefore, it is sufficient to take the two propositions (i) and (ii), or the two propositions (i) and (iii) for Postulate (II_a). That each of these decomposed parts of Postulates (II_a) is independent of the other and all the remaining postulates may be proved as follows.

Theorem $(a)_{(i)}$. *Postulate* $(II_a)_{(i)}$ *is independent of Postulates* (I_a), (I_b), $(II_a)_{(ii)}$, (II_b) and (A).

Proof. Consider the class of numerals 1, 2, 3, ..., and introduce the new laws of operations as given below.

$$m \oplus n \equiv m + n,$$

$$\begin{cases} m \otimes n \equiv n, & \text{when } m = 1, \quad (1) \\ m \otimes n \equiv 1, & \text{when } n = 1, \quad (2) \\ m \otimes n \equiv m + n - 1 + (m-1) \otimes (n-1)\,(^1), & \\ & \text{when neither of } m \text{ and } n \text{ is } 1, \quad (3) \end{cases}$$

$$(1, m) \equiv 1 \oplus m.$$

Then this class of numerals with these new laws of operations satisfies Postulates (I_a), (I_b), $(II_a)_{(ii)}$, (II_b) and (A), but not Postulate $(II_a)_{(i)}$.

1. That this class satisfies Postulates (I_a), (I_b) and (A) may be seen at once as in the case of Theorem (a).

2. It satisfies Postulates $(II_a)_{(ii)}$ and (II_b).

Firstly, from the given laws of operations, we have

$$(1.1) \otimes (1, m) \equiv 1 \otimes (1 \oplus m) \equiv 1 \otimes (1 + m) \equiv 1 + m,$$
$$(1.m) \equiv 1 \oplus m \equiv 1 + m;$$
$$\therefore\quad (1.1) \otimes (1, m) \equiv (1, m).$$

Secondly, we have

$$(1, m) \otimes (1, n) \equiv (1 \oplus m) \otimes (1 \oplus n) \equiv (1 + m) \otimes (1 + n)$$

---

(1) When both m and n are greater than 1, $m \otimes n$ is completely determined by the laws (1), (2), (3). For example,

$$\begin{aligned} 4 \otimes 3 &\equiv 4 + 3 - 1 + 3 \otimes 2 && \text{(by (3))}, \\ &\equiv (4+3-1) + (3+2-1) + 2 \otimes 1 && \text{(by (3))}, \\ &\equiv (4+3-1) + (3+2-1) + 1 && \text{(by (2))}, \\ &\equiv 11. \end{aligned}$$

$$\equiv(1+m)+(1+n)-1+m\otimes n,$$
$$\equiv 1+m+n+m\otimes n,$$
$$\{1,(m\oplus n)\oplus(m\otimes n)\}\equiv 1\oplus(m\oplus n)\oplus(m\otimes n),$$
$$\equiv 1+m+n+m\otimes n;$$
$$\therefore\quad (1,m)\otimes(1,n)\equiv\{1,(m\oplus n)\oplus(m\otimes n)\}.$$

3. It does not satisfy Postulate $(II_a)_{(i)}$.

From the given laws (1) and (2), we have

$$(1.1)\otimes(1,m)\equiv 1\otimes(1\oplus m)\equiv 1\otimes(1+m)\equiv 1+m,$$
$$(1,m)\otimes(1.1)\equiv(1\oplus m)\otimes 1\equiv(1+m)\otimes 1\equiv 1;$$
$$\therefore\quad (1.1)\otimes(1,m)\not\equiv(1,m)\otimes(1.1).$$

Theorem $(a)_{(ii)}$. *Postulate* $(II_a)_{(ii)}$ *is independent of Postulates* (I_a) (I_b), $(II_a)_{(i)}$, (II_b) and (A).

Proof. Consider the class of numerals $1, 2, 3, \ldots,$ and introduce the new laws of operations as follows:

$$m\oplus n\equiv m+n,$$
$$m\otimes n\equiv m\times n+1,$$
$$(1,m)\equiv 1\oplus m.$$

Then this class of numerals satisfies Postulates (I_a), (I_b), $(II_a)_{(i)}$, (II_b) and (A), but not Postulate $(II_a)_{(ii)}$.

1. That this class satisfies Postulates (I_a), (I_b) and (A) may be seen at once as in the case of Theorem (a).

2. It satisfies Postulates $(II_a)_{(i)}$ and (II_b).

Firstly, from the given laws of operations, we have

$$(1,m)\otimes(1.1)\equiv(1\oplus m)\otimes 1\equiv(1+m)\otimes 1,$$
$$\equiv(1+m)\times 1+1\equiv 2+m,$$
$$(1.1)\otimes(1,m)\equiv 1\otimes(1\oplus m)\equiv 1\otimes(1+m)\equiv 1\times(1+m)+1\equiv 2+m.$$
$$\therefore\quad (1,m)\otimes(1.1)\equiv(1.1)\otimes(1,m).$$

Secondly, we have

$$(1,m)\otimes(1,n)\equiv(1+m)\otimes(1+n),$$
$$\equiv(1+m)(1+n)+1\equiv 2+m+n+mn,$$
$$\{1,(m\oplus n)\oplus(m\otimes n)\}\equiv 1\oplus(m\oplus n)\oplus(m\otimes n),$$
$$\equiv 1+(m+n)+(m\times n+1),$$
$$\equiv 2+m+n+mn.$$
$$\therefore\quad (1,m)\otimes(1,n)\equiv\{1,(m\oplus n)\oplus(m\otimes n)\}.$$

3. It does not satisfy Postulate $(II_a)_{(ii)}$.

From the given law of multiplication, we have

$$(1.1)\otimes(1, m)\equiv 1\otimes(1+m)\equiv 1\times(1+m)+1\equiv 2+m,$$
$$(1, m)\equiv 1+m\,;$$
$$\therefore\ (1.1)\otimes(1, m)\not\equiv(1, m).$$

Theorem (b). Postulate (II_b) *is independent of Postulates* (I_a), (I_b) *and* (II_a).

Proof. Consider the class of numerals 1, 2, 3, ..., and introduce new laws of operations as follows.

$$m\oplus n\equiv m+n+1,$$
$$\begin{cases} m\otimes 1\equiv m\equiv 1\otimes m, \\ m\otimes n\equiv m\times n+n \text{ (when none of } m \text{ and } n \text{ is 1)}, \end{cases}$$
$$(1, m)\equiv 1\oplus m.$$

Then this class of numerals with these new laws of operations satisfies Postulates (I_a), (I_b), $II_a)$, and Postulate (A), but not Postulate (II_b).

1. It satisfies Postulate (I_a).

From the given laws of operations, we have

$$(1.1)\oplus(1, m)\equiv 1\oplus(1\oplus m)\equiv 1\oplus(1+m+1),$$
$$\equiv 1+(1+m+1)+1\equiv m+4.$$

and $$(1, 1\oplus m)\equiv 1\oplus(1\oplus m)\equiv m+4\,;$$
$$\therefore\ (1.1)\oplus(1, m)\equiv(1, 1\oplus m).$$

2. It satisfies Postulate (I_b).

$$(1\oplus 1\oplus \ldots . m \text{ times})\oplus(1\oplus 1\oplus\ldots . n \text{ times})$$
$$\equiv(m+m-1)\oplus(n+n-1),$$
$$\equiv(2m-1)\oplus(2n-1)\equiv(2m-1)+(2n-1)+1\equiv 2m+2n-1.$$
$$(1\oplus 1\oplus\ldots . m+n \text{ times})\equiv(m+n)+(m+n-1)\equiv 2m+2n-1\,;$$
$$\therefore\ (1\oplus 1\oplus\ldots .)\oplus(1\oplus 1\oplus\ldots .)\equiv(1\oplus 1\oplus\ldots .\oplus 1\oplus 1\oplus\ldots .).$$

3. It satisfies Postulate (II_a).

This is clear from the given law of operation.

4. It satisfies Postulate (A).

$$1\oplus 1\equiv 1+1+1\equiv 3\not\equiv 1.$$

5. It does not satisfy Postulate (II_b).

$$(1, m)\otimes(1, n)\equiv(1\oplus m)\otimes(1\oplus n)\equiv(2+m)\otimes(2+n),$$
$$\equiv(2+m)\times(2+n)+(2+n),$$
$$\equiv 6+2m+3n+m\times n,$$

$$\{1, (m\oplus n)\oplus(m\otimes n)\}\equiv 1\oplus\{(m+n+1)\oplus(m\times n+n)\},$$
$$\equiv 1\oplus\{(m+n+1)+(m\times n+n)+1\}$$
$$\equiv 1+\{m\times n+m+2n+2\}+1\equiv 4+m+2n+m\times n;$$
$$\therefore\quad (1, m)\otimes(1, n)\not\equiv\{1, (m\oplus n)\oplus(m\otimes n)\}.$$

Theorem (c). *Postulate* (I_a) *is independent of Postulates* (I_b), (II_a) and (II_b).

Proof. Consider the class of numerals 1, 2, 3, ..., and introduce new laws of operations as follows.

$$m\oplus n\equiv 2,$$
$$\begin{cases}1\otimes m\equiv m\equiv m\otimes 1, \\ m\otimes n\equiv 4 & \text{(when none of } m \text{ and } n \text{ is 1).}\end{cases}$$
$$(1, m)\equiv 2m.$$

Then this class of numerals with these laws of operations satisfies Postulates (I_b), (II_a), (II_b) and Postulate (A), but not Postulate (I_a).

1. It satisfies Postulate (I_b).

From the given laws of operation, we have

$$(1\oplus 1\oplus\ldots\oplus 1)\oplus(1\oplus 1\oplus\ldots\oplus 1)\equiv 2\oplus 2\equiv 2,$$
and
$$1\oplus 1\oplus\ldots\oplus 1\oplus 1\oplus 1\oplus\ldots\oplus 1\equiv 2;$$
$$\therefore\quad (1\oplus 1\oplus\ldots\oplus 1)\oplus(1\oplus 1\oplus\ldots\oplus 1)$$
$$\equiv(1\oplus 1\oplus\ldots\oplus 1\oplus 1\oplus 1\oplus\ldots\oplus 1).$$

2. It satisfies Postulate (II_a).

$$(1.1)\otimes(1, m)\equiv 1\otimes 2m\equiv 2m,$$
$$(1, m)\equiv 2m;$$
$$\therefore\quad (1.1)\otimes(1, m)\equiv(1, m).$$

Similarly we have

$$(1, m)\otimes(1.1)\equiv(1, m).$$

Thus Postulate (II_a) is also satisfied.

3. It satisfies Postulate (II_b).

$$(1, m)\otimes(1, n)\equiv 2m\otimes 2n\equiv 4 \qquad \text{(since none of } 2m \text{ and } 2n \text{ is 1),}$$
$$\{1, (m\oplus n)\oplus(m\otimes n)\}\equiv 2\{(m\oplus n)\oplus(m\otimes n)\}\equiv 2\{2\oplus(m \text{ or } n \text{ or } 4)\}$$
$$\equiv 2\times 2\equiv 4;$$
$$\therefore\quad (1, m)\otimes(1, n)\equiv\{1, (m\oplus n)\oplus(m\otimes n)\}.$$

4. It satisfies Postulate (A).

$$1 \oplus 1 = 2 \neq 1.$$

5. It does not satisfy Postulate (I_a).

$$(1.1) \oplus (1, m) = 1 \oplus 2m = 2,$$
$$(1, 1 \oplus m) = 2(1 \oplus m) = 2 \times 2 = 4\,;$$
$$\therefore\quad (1.1) \oplus (1, m) \neq (1, \mathrm{I} \oplus m).$$

Theorem (*d*). *Postulate* (I_b) *is independent of Postulates* (I_a), (II_a) and (II_b).

Proof. Consider the class of numerals 1, 2, 3, ..., and introduce new laws of operations as follows.

$$m \oplus n = 2n\,,$$
$$m \otimes n = m \times n,$$
$$(1, m) = 1 \oplus m.$$

Then this class of numbers with these laws of operations satisfies Postulates (I_a), (II_a), (II_b), and Postulate (A), but not Postulate (I_b).

1. It satisfies Postulate (I_a).

From the given laws of operations, we have

$$(1.1) \oplus (1, m) = 1 \oplus (1 \oplus m) = 1 \oplus 2m = 4m,$$
$$(1, 1 \oplus m) = 1 \oplus (1 \oplus m) = 4m\,;$$
$$\therefore\quad (1.1) \oplus (1, m) = (1, 1 \oplus m).$$

2. It satisfies Postulate (II_a).

This is clear from the above law of operation.

3. It satisfies Postulate (II_b).

$$(1, m) \otimes (1, n) = (1 \oplus m) \otimes (1 \oplus n) = 2m \otimes 2n = 2m \times 2n = 4mn,$$
$$\{1, (m \oplus n) \oplus (m \otimes n)\} = 1 \oplus \{(m \oplus n) \oplus (m \otimes n)\} = 1 \oplus \{2n \oplus mn\},$$
$$= 1 \oplus 2mn = 4\,mn.$$
$$\therefore\quad (1, m) \otimes (1, n) = \{1, (m \oplus n) \oplus (m \otimes n)\}.$$

4. It satisfies Postulate (A).

$$1 \oplus 1 = 2 \neq 1.$$

5. It does not satisfy Postulate (I_b).

For example, $(1 \oplus 1 \oplus 1) \oplus (1 \oplus 1 \oplus 1 \oplus 1) = 2 \oplus 2 = 4,$

$$(1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1) = 2.$$
$$\therefore\quad (1 \oplus 1 \oplus 1) \oplus (1 \oplus 1 \oplus 1 \oplus 1) \neq (1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1).$$

Of our two fundamental numbers $(A = A \otimes A)$, and $(B = A \oplus A)$, we

have postulated that $A \oplus A \not\equiv A$, namely $B \not\equiv A$. Under this supposition, we have constructed the numbers $(A.A)$, (A, A), (A, B), (A, C), ...; and upon these numbers, we have imposed Postulates (I) and (II) concerning the operations $\oplus$ and $\otimes$. But this postulate $A \oplus A \not\equiv A$ is not contained in Postulate (I) and (II) (though $A \otimes A \equiv A$ is contained in Postulate (II_a)). We can prove this as follows.

Theorem (e). *Postulate* (A) *is independent of Postulates* (I_a), (I_b), (II_a) and (II_b).

Proof. Consider the class of arabic numerals and define the laws of operations as given below.

$$m \oplus n \equiv n,$$
$$m \otimes n \equiv m \times n,$$
$$(1, m) \equiv 1 \oplus m.$$

Then this class of numerals satisfies Postulates (I_a), (I_b), (II_a) and (II_b), but not Postulate (A).

1. It satisfies Postulate (I_a).

$$(1.1) \oplus (1, m) \equiv 1 \oplus (1 \oplus m) \equiv 1 \oplus m \equiv m,$$
$$(1, 1 \oplus m) \equiv 1 \oplus (1 \oplus m) \equiv m;$$
$$\therefore \quad (1.1) \oplus (1, m) \equiv (1, 1 \oplus m).$$

2. It satisfies Postulate (I_b).

$$(1 \oplus 1 \oplus \ldots \oplus 1) \oplus (1 \oplus 1 \oplus \ldots \oplus 1) \equiv 1 \oplus 1 \equiv 1,$$
$$(1 \oplus 1 \oplus \ldots \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus \ldots \oplus 1) \equiv 1;$$
$$\therefore \quad (1 \oplus 1 \oplus \ldots \oplus 1) \oplus (1 \oplus 1 \oplus \ldots \oplus 1) \equiv (1 \oplus 1 \oplus \ldots \oplus 1).$$

3. It satisfies Postulates (II_a).

This is clear from the above law of operation.

4. It satisfies Postulate (II_b).

$$(1, m) \otimes (1, n) \equiv (1 \oplus m) \otimes (1 \oplus n) \equiv m \otimes n \equiv m \times n,$$
$$\begin{aligned}\{1, (m \oplus n) \oplus (m \otimes n)\} &\equiv 1 \oplus \{(m \oplus n) \oplus (m \otimes n)\} \\ &\equiv (m \oplus n) \oplus (m \otimes n) \\ &\equiv n \oplus (m \times n) \equiv m \times n;\end{aligned}$$
$$\therefore \quad (1, m) \otimes (1, n) \equiv \{1, (m \oplus n) \oplus (m \otimes n)\}.$$

5. It does not satisfy Postulate (A).

$$1 \oplus 1 \equiv 1.$$

Therefore it does not satisfy the relation $1 \oplus 1 \not\equiv 1$.

All examples given above to prove the independence of each of

Postulates (I_a). (I_b), (II_a), (II_b) from the other postulates also satisfy Postulate (A). Therefore the six postulates (A), (I_a), (I_b), $(II_a)_{(i)}$, $(II_a)_{(ii)}$, (II_b) are independent of one another.

System of Numbers as a Set of Things.

As a set of things, this system of numbers has the following properties:

1. Property of simply ordered class;
2. Property of discrete class;
3. Property of denumerable class;
4. Property of class having the first, but no last element.

Namely, this class is a simplest one of denumerable classes.

Conclusion. Here we shall recapitulate briefly our mode of constructing natural numbers.

Starting with only one thing A, and operating this thing A to itself, we get two different things by two different operations then used. One of these things is identical with A itself while the other is different from A. Next taking these two things in pair, we define the third things; and further taking the first and the third things in pair, we define the fourth thing; and taking the first and the fourth things in pair we define the fifth thing; and so on; all these obeying one and the same mode of combination. The class of all things thus constructed are called a system of natural numbers. In order to treat them as numbers, we imposed upon them Postulate (I) and (II) for two operations, addition and multiplication, and Definitions (A) and (B) for two relations, equality and inequality. By means of these postulates and definitions, we proved the following:

1. *All theorems concerning so called natural numbers are true in our system;*
2. *Our postulates are independent of one another;*
3. *Our number-system is free from contradiction;*
4. *Two direct operations* $\oplus$ *and* $\otimes$ *can be performed without any restriction while corresponding two inverse operations* $\ominus$ *and* $\oplus\!\!\!\!\div$ *can only be performed under certain condition;*
5. *There exists one and only one unit-element, but none of zero-element in our system;*
6. *As a set of things, this system is a simplest denumerable class; in other words, it forms a discrete series having the first, but no last element.*

(B) Integers.

Introduction of New Numbers.

Consider the class of natural numbers already defined, and take any two numbers of them, and denote them by a, b. *From this pair of numbers, we construct a new thing which we denote by* (a, b). In order that we may treat these newly constructed things as numbers, or in other words, in order that we may compare them and operate them, we lay down the following definitions and postulates.

Definition (A). The two numbers (a, b) and (a', b') are said to be equal when and only when the relation $a \oplus b' = a' \oplus b$ holds good.

Definition (B). The number (a, b) is said to be greater than or less than the number (a', b') according as $a \oplus b'$ is greater than or less than $a' \oplus b$.

Remark. Since a, b, a', b' are natural numbers, the meaning of the operations $\oplus, \otimes$, and the symbols $>, =, <$ are already known, so that $a \oplus b' = a' \oplus b$, $a \oplus b' \gtrless a' \oplus b$ have definite meanings.

Postulate (I). $(a, b) \oplus (a', b') \equiv (a \oplus a', b \oplus b')$.

Postulate (II). $(a, b) \otimes (a', b') \equiv (aa' \oplus bb', ab' \oplus ba')$.

Remark. aa' is used for the abbreviation of $a \otimes a'$. Since a, b, a', b' are natural numbers, $a \oplus a', aa' \oplus bb', \ldots$ are also known natural numbers, so that $(a \oplus a', b \oplus b')$ and $(aa' \oplus bb', ab' \oplus ba')$ may be denoted by (m, n) and (p, q) respectively. Therefore they are elements of the system of new numbers. Thus from these postulates, we have the following fundamental theorem.

If (a, b) and (a', b') are any two elements of the new system of numbers, then both $\{(a, b) \oplus (a', b')\}$ and $\{(a, b) \otimes (a', b')\}$ are also elements of this system.

The class of all numbers $\{(a, b)\}$ with the above laws of operations and the above definitions of equality and inequality is called the class of integers.

Here we shall study what numbers are defined by the above postulates and definitions.

Theorem 1. The new system of numbers is a more extended one than the system of natural numbers and contains the latter system as its subclass.

Proof. (1). First of all, when $a > b$, the new number (a, b) is equivalent to the natural number $a \ominus b$. For, (a) when $a > b$ and $a' > b'$, if the relation $(a, b) = (a', b')$ holds good, then, by the definition of

equality, we have the relation $a \oplus b' = a' \oplus b$; and therefore, by Theorem 16, we have the relation $a \ominus b = a' \ominus b'$; and conversely we may easily prove that, if the relation $a \ominus b = a' \ominus b'$ holds good, then also the relation $(a, b) = (a', b')$ holds good. (b) Similarly, it may be proved that, when $a > b$ and $a' > b'$, if the relation $(a, b) \gtreqless (a', b')$ holds good, then the relation $(a \ominus b) \gtreqless (a' \ominus b')$ also holds good, and conversely, if the relation $(a \ominus b) \gtreqless (a' \ominus b')$ holds good, then the relation $(a, b) \gtreqless (a', b')$ also holds good. (c) Further, if we put $a \ominus b$ instead of (a, b) in Postulates (I) and (II), then the postulates are all satisfied. Thus all definitions and postulates which completely define the new number (a, b) are all satisfied by the natural number $a \ominus b$ when $a > b$. Therefore our new number may be considered to represent the natural number when $a > b$. Moreover, it may easily be seen that any natural number may be represented by one of our new numbers which has the form $(a, b)_{a>b}$ Hence we may say that the class of all natural numbers may be represented by the subclass of new numbers, which we denotes by $\{(a, b)_{a>b}\}$.

(2) Secondly, when $a = b$, the new number (a, b) defines a zero-element of the system. For, by Postulate (I), we have

$$(a, b) \oplus (a, b) = (a \oplus a, b \oplus b) = (2a, 2b).$$

But, by hypothesis, a is equal to b, and so by the definition of equality of natural number, a is identical with b; therefore we have

$$(a, a) \oplus (a, a) = (2a, 2a).$$

Now, by the definition of equality of the new numbers, we have

$$(a, a) = (2a, 2a) = (na, na),$$

since $a \oplus 2a = 2a \oplus a$ and $a \oplus na = na \oplus a$ are true.

Accordingly we have the relation

$$(a, a) \oplus (a, a) = (2a, 2a) = (a, a),$$

so that (a, a) is a zero-element by the definition. Thus the number $(a, b)_{a=b}$ is a new number not contained in the class of natural numbers.

(3) Thirdly, when $a < b$, the number (a, b) has no corresponding one in the class of natural numbers, nor it is a zero-element, so the assemblage of them defines a quite new system of numbers. We shall call this class of numbers the class of negative integers. In contrast to this, we shall call the class of natural numbers the class of positive integers. Thus the assemblage of all positive and negative integers and

zero-element constitutes the class of integers.

Fundamental Properties of the New System of Numbers.

From the above definitions and postulates, it may be proved that all fundamental theorems and accordingly all theorems concerning the operations $\oplus$ and $\otimes$ hold good in this system of numbers. Since the proof may be effected in a similar manner as in the case of natural number, we shall not enter into it. But here we shall give the proof of the proposition $(a, b)\,c = c(a, b) = (ac, bc)$ only, in order to show that, though it is taken as postulate by some writers, it does not need to do so in our mode of treatment.

Now, by hypothesis, c is a natural number, therefore it may be denoted by $m \ominus n (m > n)$ (for, taking any natural number n, we can always find a natural number m, such that $n \oplus c = m$, by the fundamental property of our natural number; and hence $c = m \ominus n$). Accordingly c may be denoted by $(m, n)_{m>n}$ and we have

$$\begin{aligned}(a, b)\,c &= (a, b) \otimes (m, n) = (am \oplus bn, an \oplus bm) &&\text{(by Postulate } (II)),\\ &= \{a(m \ominus n), b(m \ominus n)\} &&\text{(by Definition } (A)),\\ &= (ac, bc).\end{aligned}$$

Removal of Restriction of Inverse Operation.

Theorem 2 Two direct operations (addition and multiplication) can be performed without any restriction in the system of integers.

This follows at once from Postulates (I) and (II).

In the system of natural numbers, the inverse operation $a \ominus b$ is possible only when $a > b$; in all other cases it is impossible. But, in the new system of numbers, this restriction is removed, and the inverse operation $\ominus$ is always possible for any two numbers of the system. That is, we may prove the following theorem:

Theorem 3. The first inverse operation (subtraction) can be performed without any restriction in the system of integers.

Proof. Taking any two numbers (a, b) and (a', b') of the system, we have to find the number (u, v) satisfying the relation

$$(a, b) \oplus (u, v) = (a', b')$$

in order to prove the theorem. Now, by Postulate (I), we have the relation

$$(a, b) \oplus (u, v) = (a \oplus u, b \oplus v).$$

Therefore, we have to find the natural numbers u, v, satisfying the relation

$$(a \oplus u, b \oplus v) = (a', b'),$$

or $\quad a \oplus u \oplus b' = b \oplus v \oplus a' \quad$ (by definition of equality),

or $\quad u \oplus (a \oplus b') = (b \oplus a') \oplus v \quad$ (by property of natural numbers).

To satisfy this condition, it is sufficient to take (u, v), such that the relation

$$(u, v) = (b \oplus a', a \oplus b')$$

holds good. Now $b \oplus a'$ and $a \oplus b'$ are both natural numbers, therefore $(b \oplus a', a \oplus b')$ is a number belonging to our new system of numbers.

Theorem 4. The second inverse operation (division) cannot be performed without any restriction in the system of integers.

In order that the division may be performed without any restriction, it is necessary that for any two given numbers (a, b) and (a', b'), we can always find two natural numbers u, v, such that the relation

$$(a, b) \otimes (u, v) = (a', b'),$$

or
$$(au \oplus bv, av \oplus bu) = (a', b'),$$

or
$$au \oplus bv \oplus b' = av \oplus bu \oplus a' \qquad (a)$$

holds good. But, when the two numbers $(3, 1)$ and $(2, 1)$ are taken for (a, b) and (a', b') respectively, the last relation (a) becomes

$$2u = 2v \oplus 1\,;$$

and this relation cannot be satisfied whatever natural numbers are taken for $u. v$, since $2u$ is a multiple of 2 while $2v \oplus 1$ is not a multiple of 2.

Existence of Particular Elements.

Theorem 5. There exists one and only one zero-element in the system of integers.

That (a, a) is a zero-element was proved before, and that there is no other zero-element may be proved in a similar manner as in Theorem 18.

The first fundamental property of zero-element, "the product of any number and zero is always equal to zero," may easily be seen to be true in this system of numbers, since we have the relation

$$(a, a) \otimes (p, q) = (ap \oplus aq, aq \oplus ap) = (ap \oplus aq, ap \oplus aq).$$

The second fundamental property of zero-element, "if (a, b) is equal to (a', b'), then $(a, b) \ominus (a', b')$ is equal to zero," may be proved as follows.

By Theorem 3, we have

$$(a, b) \ominus (a', b') = (u, v) = (b' \oplus a, a' \oplus b),$$

and by Postulate (I), we have

$$(b' \oplus a, a' \oplus b) \oplus (b' \oplus a, a' \oplus b) = \{(b' \oplus a) \oplus (b' \oplus a), (a' \oplus b) \oplus (a' \oplus b)\},$$
$$= \{2(b' \oplus a), 2(a' \oplus b)\}.$$

But, from the hypothesis $(a, b) = (a', b')$, we have

$$a \oplus b' = b \oplus a'.$$

Therefore, from the definition of equality, we have

$$\{2(b' \oplus a), 2(a' \oplus b)\} = (b' \oplus a, a' \oplus b),$$

and accordingly, we have

$$(b' \oplus a, a' \oplus b) \oplus (b' \oplus a, a' \oplus b) = (b' \oplus a, a' \oplus b),$$

which shows that $(b' \oplus a, a' \oplus b)$ and accordingly $(a, b) \ominus (a', b')$ is a zero-element. We shall denote this property in the symbolical form, "if $(a, b) = (a', b')$, then $(a, b) \ominus (a', b') = 0$."

The converse of this theorem is also true.

In this system of numbers, the relation $(a, b) \oplus (b, a) = 0$ always holds good whatever a and b may be. In this case, (b, a) is called an opposite element of (a, b) and is denoted by the symbol $-(a, b)$.

Theorem 6. There exists one and only one unit-element in this system of numbers.

Proof. By Postulate (II), we have

$$\begin{aligned}(a \oplus 1, a) \otimes (a \oplus 1, a) &= [\{(a \oplus 1) \otimes (a \oplus 1)\} \oplus (a \otimes a),\\ &\qquad \{(a \oplus 1) \otimes a\} \oplus \{a \otimes (a \oplus 1)\}]\\ &= \{(aa \oplus 2a \oplus 1) \oplus aa, (aa \oplus a) \oplus (aa \oplus a)\}\\ &= (2aa \oplus 2a \oplus 1, 2aa \oplus 2a),\end{aligned}$$

but, by the definition of equality, we have

$$(a \oplus 1, a) = (2aa \oplus 2a \oplus 1, 2aa \oplus 2a).$$

Therefore, we have

$$(a \oplus 1, a) \otimes (a \oplus 1, a) = (a \oplus 1, a).$$

Accordingly, by the definition of unit-element, $(a \oplus 1, a)$ is a unit-element of the system. Now a may be any natural number, but $(a_1 \oplus 1, a_1)$ and $(a_2 \oplus 1, a_2)$ are equal to each other by the definition of equality of our numbers.

Further, we may prove that there is no unit-element other than $(a \oplus 1, a)$. For, any unit element (p, q) must satisfy the relation

$$(p, q) \otimes (p, q) = (p, q),$$

or
$$(pp \oplus qq, pq \oplus qp) = (p, q),$$

or
$$pp \oplus qq \oplus q = 2pq \oplus p.$$

or
$$p = q \text{ or } p = q \oplus 1.$$

But (p, q) is a zero-element when $p = q$; therefore any unit-element (if any) must be of the form $(q \oplus 1, q)$. Therefore there is only one unit-element $(a \oplus 1, a)$,

Definition. If a number A, which is not a unit-element, satisfy the relation $A \otimes A = 1$, then A is called the negative unit-element of the number-system considered. In contrast to the negative unit-element, the ordinary unit-element is called the positive unit-element.

Theorem 7. There exists one and only one negative unit-element in our system of numbers.

That $(b \ominus 1, b)$ is a negative unit-element and that there is no other negative unit-element may be proved in a similar manner as in the previous theorem.

From our postulates and definitions, we may easily prove the following theorem.

Theorem 8. (1) All integers may be produced from natural numbers by multiplying them to the positive and negative unit-elements.

(2) The sum of positive and negative unit-element is equal to zero.

(3) The product of any number (p, q) and the positive unit-element is equal to the number (p, q).

Non-Contradiction and Independence of Postulates.

Postulates (I), (II), and Definitions (A), (B) are all satisfied by any number (a, b) of the system, whether a is equal to, or greater than, or less than b. Therefore, the results deduced by applying pure logic to these postulates and definitions must hold good for any number of the system. Now, among these results, there cannot occur any contradiction. For, since the above postulates and definitions are all satisfied by the system of natural numbers as was already pointed out, and moreover, since the system of natural numbers implies no contradiction as was already proved, if there would arise a contradiction in the logical combination of Postulates (I), (II) and Definitions (A), (B), then there would also arise a contradiction in the system of natural numbers.

Further, Postulate (I) defines the operation of addition, and Postulate (II) defines another operation called multiplication; and there is only one postulate for each of the two different operations. Moreover, neither of them presupposes the other. Therefore, that they are independent of each other will be evident and will not need to be discussed in detail.

System of Numbers as a Set of Things.

As a set of things, this system of numbers has the following properties:

1. Property of simply ordered class;
2. Property of denumerable class;
3. Property of discrete class;
4. Property of class having no first and no last elements.

Namely the system of numbers is an unlimited discrete series.

Thus, from the system of natural numbers, we have constructed a more extended system of numbers which satisfies the four great principles:

1. Principle of permanency of form,
2. Principle of freedom of direct and inverse operations of the first order and a direct operation of the second order.
3. Principle of non-contradiction,
4. Principle of non-density.

(*To be continued.*)

On Multiplicative and Enumerative Properties of Numerical Functions,

by

Echo D. Pepper, Seattle, Wash., U. S. A.

A method is presented here by which an extensive class of theorems appearing in Dickson, History of the Theory of Numbers, (Carnegie Institution of Washington, 1919), Chapters V and X, hitherto obtained individually, may be derived by direct analysis.

The method is discussed in Section I and is followed by the derivation, in Section II of many of the more familiar theorems. Section III contains the generalization of the same method to the realm of complex integers; Section IV the extension to ideals, with the derivation of theorems analogous to those in Section II. Section V contains the extension to polynomials with respect to a prime ideal modulus, and prime modulus of any realm.

I. Theory of the numerical functions in the rational realm.

1. A function $f(n)$ is called numerical, if for each integral value $\geqq 0$ of n, $f(n)$ takes a single definite value. Thus, Euler's $\phi(n)$, the totient of n, and $\delta(n)$, the number of divisors of n, are numerical. We shall consider a certain class of theorems concerning such functions.

2. We shall first determine the infinite products which generate certain functions, such products being called "generators," and by use of these only, derive relations among these functions. The method will also be generalized for the realm $k(i)$, for ideals, and for polynomials to a prime ideal modulus, also to a rational prime modulus.

3. The determination of a product will be clear from an example. Let, as usual, $J_k(n)$ or $\phi_k(n)$, denote the Jordan totient of index k, viz., the number of sets of k (equal or distinct) positive integers $\leqq n$, such that the greatest common divisor of the integers of each set is prime to n, two sets being different if the order of the integers in them be different.

Let, as always henceforth,

$n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot p_3^{\alpha_3}\cdots\cdot p_r^{\alpha_r}$ be the prime resolution of n.

Then $$\phi_k(n)=\phi_k(p_1^{\alpha_1})\phi_k(p_2^{\alpha_2})\cdots\cdots\phi_k(p_r^{\alpha_r}).$$

Consider the infinite product, in which $p_i(i=1,\ 2,\ 3,\ \cdots)$ are all distinct primes

$$(1)\quad \Pi_i\left[1+\frac{\phi_k(p_i)}{p_i^s}+\frac{\phi_k(p_i^2)}{p_i^{2s}}+\frac{\phi_k(p_i^3)}{p_i^{3s}}+\cdots\cdots+\frac{\phi_k(p_i^{\alpha_i})}{p_i^{\alpha_i s}}+\cdots\cdots\right],$$

which can easily be shown absolutely convergent for suitably chosen $|s|$. Assure such an $|s|$ in what follows. The general term of the expansion of (1) is:

$$\frac{\phi_k(p_i^{\alpha_1})\cdot\phi_k(p_i^{\alpha_2})\cdots\cdot\phi_k(p_i^{\alpha_r})}{p_i^{\alpha_1 s}\cdot p_i^{\alpha_2 s}\cdots\cdot p_i^{\alpha_r s}}=\frac{\phi_k(n)}{n^s},$$

since one and only one term must be taken from each factor of Π_i in constructing each term of the product.

Therefore

$$\Pi_i\left[1+\frac{\phi_k(p_i)}{p_i^s}+\frac{\phi_k(p_i^2)}{p_i^{2s}}+\frac{\phi_k(p_i^3)}{p_i^{3s}}+\cdots\cdots+\frac{\phi_k(p_i^{\alpha_i})}{p_i^{\alpha_i s}}+\cdots\cdots\right]=\sum_{n=1}^{\infty}\frac{\phi_k(n)}{n^s},$$

since $\phi_k(p^r)=p^{rk}-p^{(r-1)k}$, this reduces to:

$$\Pi_i\left[1+\frac{p_i^k-1}{p_i^s}+\frac{p_i^{2k}-p_i^k}{p_i^{2s}}+\cdots\cdots+\frac{p_i^{\alpha_i k}-p_i^{(\alpha_i-1)k}}{p_i^{\alpha_i s}}+\cdots\cdots\right]$$

$$=\Pi_i\left[\frac{1-\dfrac{1}{p_i^s}}{1-\dfrac{p_i^k}{p_i^s}}\right].$$

We then say, $\left[\dfrac{1-\dfrac{1}{p_i^s}}{1-\dfrac{p_i^k}{p_i^s}}\right]$ is the *generator* of $\phi_k(n)$, where the [ ] is a convenient notation for the infinite product, of which the typical factor is given in the brackets.

4. Before proceeding further, a shorter notation will be introduced.

(i.) Let $p^{-s}=z$ where p is an arbitrary prime, and s is chosen so that all the series and products in which it appears are absolutely convergent.

(ii.) Let $G'(z)=\Pi_p[1+f_1(p)z^{\alpha_1}+f_2(p)z^{\alpha_2}+\cdots\cdots+f_r(p)z^{\alpha_r}+\cdots\cdots]$ be an infinite product, in which $\alpha_1,\alpha_2,\ \cdots\ \alpha_r,\ \cdots$ are integers >0, the $f_i(p)$ $[i=1, 2,\cdots]$ are arbitrary functions of p, and the p in p^{-s} is con-

sidered inseparable from p^{-s}, viz., p^{-s} is the parameter z, and we do not have $pz^s = z^{s-1}$.

(iii.) When $G'(z)$ is expanded, a Dirichlet series of the form $\sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is obtained. We then say that $G'(z)$ is the *generator* of $g(n)$, and write it symbolically $G'(z) \cdot\!F\; g(n)$, which is to be read, $G'(z)$ is the generator of $g\ (n)$.

(iv.) Thus, suppose $G'(z) \cdot\!F\; g\ (n)$ and $H'\ (z) \cdot\!F\; h(n)$, it is required to find the function generated by $G'(z).\ H'\ (z)$.

Now $G'(z) \cdot\!F\; g(n)$ implies $G'(z) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$, and $H'(z) \cdot\!F\; h(n)$ implies

$H'(z) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$, whence

$$(1) \quad G'(z) \cdot H'(z) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{g(n_1) \cdot h(n_2)}{(n_1 n_2)^s}.$$

Let $n_1 n_2 = n$, then since n, n_1, n_2, are all integers >0, n_1 is any divisor of n, and n_2 is it's conjugate divisor. Further, let $\Sigma_n f_1(d) f_2(\delta)$ mean that the sum extends to all pairs of divisors d, δ, such that $d\ \delta = n$. Then replacing n_1 by d, n_2 by δ, and $n_1 n_2$ by $d\delta = n$, we have from (1)

$$G'(z) \cdot H'(z) = \sum_{n=1}^{\infty} \frac{\Sigma_n g(d) h(\delta)}{n^s}.$$

Therefore we say $G'(z) \cdot H'(z) \cdot\!F\; g(d)h(\delta)$, viz, the formal *algebraic* product of $G'(z)$ and $H'(z)$ generates the *symbolic* product $g(d)h(\delta)$ or $\Sigma_n g(d)h(\delta)$ of $g(n)$ and $h(n)$.

(v.) Further, denote by $G(z)$ the general factor

$$1 + f_1(p) z^{\alpha_1} + f_2(p) z^{\alpha_2} + \cdots\cdots + f_r(p) z^{\alpha_r} + \cdots\cdots$$

of the infinite product $G'(z) \equiv [1 + f_1(p) z^{\alpha_1} + f_2(p) z^{\alpha_2} + \cdots\cdots + f_r(p) z^{\alpha_r} + \cdots\cdots]$. We then call $G(z)$ the *associate* of g, and write it symbolically $G\ (z) \sim g$ [$\sim$ means, "is associate of"].

(vi.) In the summation $\Sigma_n g(d)h(\delta)$, drop the suffix n, *it being understood that* $\Sigma g(d)h(\delta)$ *refers to all pairs* (d, δ) *of conjugate divisors of an arbitrary integer* $n > 0$; further, drop the d, δ and the summation sign Σ, so that we write simply $g\ h$ for $\Sigma_n g(d)h(\delta)$. Also, wherever $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$

appears, it will be written $\Sigma\frac{f(n)}{n^s}$, the summation always extending over all values $1\leqq n\leqq\infty$.

(vii.) We may then write symbolically: if $G(z)\sim g$, and $H(z)\sim h$ and $G(z)H(z)=K(z)\sim k$ then $gh=k$. Hence we may rewrite the conclusion in 3., in this notation, as follows: $\left[\frac{1-z}{1-p^kz}\right]\mathcal{F}\,\phi\,(n)$ or more briefly $\frac{1-z}{1-p^kz}\sim\phi^k$. Note that in the latter expression the [ ] indicating an infinite product have been dropped, viz., the associate of ϕ^k is $\frac{1-z}{1-p^kz}$.

5. Using this notation, let us find the product which generates the function $n^\mu\sigma_\mu(n)$ [$\sigma_\mu(n)$=sum of $\mu\,th$ powers of the divisors of n].

Let $n=\Pi_i\,p_i^{\alpha_i}$, then $n^\mu=\Pi_i p_i^{\mu\alpha_i}$, and $\sigma_\mu(n)=\Pi_i\,\frac{p_i^{\mu(\alpha_i+1)}-1}{p_i^\mu-1}$,

whence $$n^\mu\sigma_\mu(n)=\Pi_i\,p_i^{\mu\alpha_i}\frac{p_i^{\mu(\alpha_i+1)}-1}{p_i^\mu-1}.$$

Consider the sum:

$$1+\Sigma\,p^{\mu\alpha}\frac{p^{\mu(\alpha+1)}-1}{p^\mu-1}Z$$
$$=1+\Sigma(p^{2\mu\alpha}+p^{2\mu\alpha-\mu}+p^{2\mu\alpha-2\mu}+\cdots\cdots+p^{\mu\alpha})Z^\alpha,$$

which for all values of the integer $\alpha>0$, gives

$$1+(p^{2\mu}-p^\mu)Z+(p^{4\mu}+p^{3\mu}+p^{2\mu})Z^2+\cdots\cdots\cdots\cdots$$
$$=\frac{1}{(1-p^\mu z)(1-p^{2\mu}z)}.$$

Following the notation of 4., we may write

$$\left[\frac{1}{(1-p^\mu z)(1-p^{2\mu}z)}\right]\mathcal{F}\;n^\mu\sigma_\mu(n),\text{ or}$$

$$\frac{1}{(1-p^\mu z)(1-p^{2\mu}z)}\sim u_\mu\sigma_\mu[n^\mu=u_\mu\text{ or all }n,\ 9.04].$$

6. As an example of the reverse process, that of finding the function, which a given product generates; consider the generator $\left[\frac{1}{(1-pz)^2}\right]$.

Now $\left[\frac{1}{(1-pz)^2}\right]=\left[\frac{1}{1-pz}\cdot\frac{1}{1-pz}\right]=u_1(n)\cdot u_1(n)$, [where $u_1(n)=n$ for all n, see 9.03], or written as a summation

$$=\sum\frac{u_1(n)}{n^s}\cdot\sum\frac{u_1(n)}{n^s}=\sum\frac{\Sigma u_1(d)u_1(\delta)}{n^s}=\sum\frac{\Sigma d\delta}{n^s}.$$

Obviously $\Sigma d\delta=n\tau(n)$, [$\tau(n)$=number of divisors of n].

Hence $$\left[\frac{1}{(1-pz)^2}\right]\mathcal{F}\ n\tau(n),\text{ or }\frac{1}{(1-pz)^2}\sim(u_1\tau).$$

7. We shall now define units and reciprocals. If $\varepsilon(1)=1$ and $\varepsilon(n)=0$ for all $n>1$, then $\varepsilon(n)$ is defined as the *unit function*, called ε, and it's associate is 1, viz., $1\sim\varepsilon$.

Let $G(z)\sim g$ and $\frac{1}{G(z)}\sim g'$, then $G(z)\cdot\frac{1}{G(z)}=1\sim\varepsilon$.

Hence $$gg'=\varepsilon\ [i.\ e.\ \sum_n g(d)g'(\delta)=1].$$

We then say G and G' are *reciprocals* of each other. Thus, we have $\frac{1}{1+z}\sim\lambda$ [see 10.14] and $1+z\sim(\mu^2)$ [see 10.02], where (μ^2) is the symbolic notation for $\{\mu(n)\}^2\equiv\mu(n)\,\mu(n)$. This is not to be confused with symbol μ^2, written without ( ), which represents $\Sigma_n\mu(d)\mu(\delta)$ and not $\Sigma\{\mu(d)\}^2$. Then, $\frac{1}{1+z}\cdot 1+z\sim\varepsilon$, hence $(\mu^2)\lambda=\varepsilon$, viz., (μ^2) and λ are reciprocals of each other.

We shall have occasion to use some compound functions which are combinations of simple function and not simply products of them. Thus $(u_\mu\sigma_\mu)$ represents $\Sigma u_\mu(n)\sigma_\mu(n)$, while $u_\mu\sigma_\mu$, written without ( ), and therefore a symbolic product, represents $\Sigma u_\mu(d)\sigma_\mu(\delta)$.

8. A detailed proof of one theorem will be sufficient to illustrate the method used. Let us prove a formula of Liouville taken from Dickson, History of the Theory of Numbers, Vol. I. p. 286,

$$\Sigma d^\mu\sigma_\mu(d)=\Sigma\delta^{2\mu}\sigma_\mu(d).$$

Now $$\frac{1}{(1-p^\mu z)(1-p^{2\mu}z)}\sim(u_\mu\sigma_\mu),\ [u_\mu=n^\mu\text{ for all }n,\ 9.04,\ 10.09]$$

and $$\frac{1}{(1-p^\mu z)(1-p^{2\mu}z)}\cdot\frac{1}{1-z}\sim(u_\mu\sigma_\mu)u_0\ [\text{see }10.03]$$

then, rearranging the factors, we have

$$\frac{1}{(1-p^{2\mu}z)}\cdot\frac{1}{(1-p^\mu z)(1-z)}\sim u_{2\mu}\cdot\sigma_\mu\ [\text{see }10.06,\ 10.12].$$

Whence $(u_\mu\sigma_\mu)u_0=u_{2\mu}\cdot\sigma_\mu$, or $(u_\mu\sigma_\mu)=u_{2\mu}\cdot\sigma_\mu$, [$u_0=1$ for all n, 9.02]

which according to the notation adopted in 4., is a symbolic representation of $\Sigma d^{\mu}\sigma_{\mu}(d) = \Sigma \delta^{2\mu}\sigma_{\mu}(d)$.

9. A list of certain functions occurring in Dickson, History of the Theory of Numbers, Vol. I. Chapters V, X, will be put here for reference.

.01 $\mu(n)=0$, if n be divisible by a square >1; $=+1$, if n is the product of an even number of primes; $=-1$, if n is the product of an odd number of primes, Möbius' function.

.02 $u_0(n)=1$, for all values of n, Riemann's ζ-function.

.03 $u_1(n)=n$, for all values of n.

.04 $u_{\mu}(n)=n^{\mu}$, for all values of n.

.05 $\tau(n)=\nu(n)$, the number of divisors of n.

.06 $\sigma(n)$, the sum of the divisors of n.

.07 $\sigma_{\mu}(n)$, sum of the μth powers of the divisors of n.

.08 $\lambda(n)=(-1)^{\pi(n)}$, $\pi(n)$ represents the total number of divisors of n.

.09 $k_{\mu}(n)=$ $+1$, if n a perfect μth power; $=0$ otherwise.

.10 $\theta(n)$, number of resolutions of n into two relatively prime factors.

.11 $\phi(n)$, numbers prime to n, and not greater than n, Euler's ϕ-function.

.12 $\varepsilon(n)$, unit, $=+1$ if $n=1$; $=0$ for all other n.

10. The generators of the above functions, together with the generators of some compound functions will be put here.

.01	$\mu(n)$	$1-z$	.02	$\{\mu(n)\}^2$	$1+z$	.03	$u_0(n)$	$\dfrac{1}{1-z}$

.04	$u_1(n)$	$\dfrac{1}{1-pz}$	.05	$n^{\mu}\sigma_{\nu+t}(n)$	$\dfrac{1}{(1-p^{\mu}z)(1-p^{\mu+\nu+t}z)}$
.06	$u_{\mu}(n)$	$\dfrac{1}{1-p^{\mu}z}$	.07	$n^{\mu}\sigma_{2\mu}(n)$	$\dfrac{1}{(1-p^{\mu}z)(1-p^{3\mu}z)}$
.08	$\tau(n)$	$\dfrac{1}{(1-z)^2}$	.09	$n^{\mu}\sigma_{\mu}(n)$	$\dfrac{1}{(1-p^{\mu}z)(1-p^{2\mu}z)}$
.10	$\sigma(n)$	$\dfrac{1}{(1-z)(1-p^2z)}$	.11	$n\sigma(n)$	$\dfrac{1}{(1-pz)(1-p^2z)}$
.12	$\sigma_{\mu}(n)$	$\dfrac{1}{(1-z)(1-p^{\mu}z)}$	.13	$n\lambda(n)$	$\dfrac{1}{1+pz}$
.14	$\lambda(n)$	$\dfrac{1}{1+z}$	.15	$n\sigma_{\mu-1}(n)$	$\dfrac{1}{(1-pz)(1-p^{\mu}z)}$
.16	$k_{\mu}(n)$	$\dfrac{1}{1-z^{\mu}}$	.17	$\lambda(n)\sigma(n)$	$\dfrac{1}{(1+z)(1+pz)}$

.18	$\theta(n)$	$\dfrac{1+z}{1-z}$	.19	$\lambda(n)\theta(n)$	$\dfrac{1-z}{1+z}$
.20	$\phi(n)$	$\dfrac{1-z}{1-pz}$	.21	$\{\theta(n)\}^{\mu}$	$\dfrac{1+(2^{\mu}-1)z}{1-z}$
.22	$\phi_k(n)$	$\dfrac{1-z}{1-p^k z}$	.23	$\lambda(n)\tau(n^2)$	$\dfrac{1-z^2}{(1+z)^3}$
.24	$\varepsilon(n)$	1	.25	$\lambda(n)\tau(n)$	$\dfrac{1}{(1+z)^2}$
.26	$\tau(n^2)$	$\dfrac{1+z}{(1-z)^2}$	.27	$\{\tau(n)\}^2$	$\dfrac{1+z}{(1-z)^3}$
.28	$n\tau(n)$	$\dfrac{1}{(1-pz)^2}$	.29	$\tau(n^{\mu})$	$\dfrac{1+(\mu-1)z}{(1-z)^2}$
.30	$n^{\mu}\tau(n)$	$\dfrac{1}{(1-p^{\mu}z)^2}$	.31	$\tau(n)\tau(n^{\mu})$	$\dfrac{1+(2\mu-1)z}{(1-z)^3}$
.32	$\tau(n^{2\mu})$	$\dfrac{1+(2^{\mu}-1)z}{(1-z)^2}.$			

II. Relations between functions in the rational realm.

1. To illustrate the use of such a list, let us derive some relations among the functions there represented. Taking the generators from I, 10 and proceeding as in I, 8, we obtain the following relations.

$u_0\mu=\varepsilon$, $u_1u_0=\sigma$, $u_1u_0=\phi\tau$, $\mu u_1=\phi$, $u_0(\mu^2)=\theta$, $\theta\tau=(\tau^2)$, $u_1\tau=\sigma u_0$, $u_{\mu}u_0=\sigma_{\mu}$, $\lambda\theta=u_0$, $\phi u_0=u_1$, $(\lambda\theta)\theta=\varepsilon$, $(\lambda\theta)u_0=\lambda$, $(\tau)^2(\lambda\theta)=\tau$, $(u_{\mu}\sigma_{\mu})u_0=\sigma_{\mu}u_{2\mu}$, $(\lambda\theta)\tau=K_2$, $u_{\mu}\sigma_v=u_v\sigma_{\mu}$, $(u_{\mu}\sigma_{v+t})\sigma_v=(u_v\sigma_{\mu+t})\sigma_{\mu}$, $(\mu^2)\lambda=\varepsilon_1\mu\tau=u_0$, $u_{\mu}\sigma_{3\mu}=u_0(u_{\mu}\sigma_{2\mu})$, $\phi\tau=\sigma$, $(u_1\sigma)u_0=u_2\sigma$, $(u_{\mu}\tau)u_0=u_{\mu}\sigma_{\mu}$, $\Sigma\tau(\delta^2)\lambda(d)=\tau(n)$, $\Sigma d\tau(\delta^2)=\Sigma\theta(\delta)\sigma(d)$, $\Sigma\phi(d)=n$, $\Sigma\phi_k(d)=n^k$, $\Sigma\theta(d)=\tau(n^2)$, $\Sigma d\sigma(d)=\Sigma\delta^2\sigma(d)$, $\Sigma\theta(\delta)\tau(d)\tau(d^{\mu})=\Sigma\tau(\delta^2)\tau(d^{\mu})$, $\Sigma d\tau(d)=\Sigma\delta\sigma(d)$, $\Sigma d\tau(d)\tau(\delta)=\Sigma\sigma(d)\sigma(\delta)$, $\Sigma\tau(d^{2\mu})\theta(\delta)=\Sigma\{\theta(d)\}^{\mu}\tau(\delta^2)$, $\Sigma\phi(d)\sigma(\delta)=n\tau(n)$, $\Sigma\phi(\delta)\tau(d^{2\mu})=\Sigma\delta\{\theta(d)\}^{\mu}$, $\Sigma\lambda(d)\theta(d)\tau(\delta^2)=u_0(n)=1$, $\Sigma\tau(d^{2\mu})\tau(\delta)=\Sigma\tau(d)\tau(d^{\mu})$, $\Sigma\tau(d^{2\mu})=\tau(n)\tau(n^{\mu})$, $\Sigma\tau(\delta^{2\mu})\sigma(d)=\Sigma\tau(d)\tau(d^{\mu})\delta$, $\Sigma\tau(\delta^2)\phi(d)=\Sigma\delta\theta(d)$, $\Sigma\phi(d)\tau(\delta)\tau(\delta^{\mu})=\Sigma d\tau(\delta^{2\mu})$, $\Sigma\{\theta(d)\}=\tau(n^{2\mu})$, $\Sigma\lambda(d)\tau(d^2)\sigma_{\mu}(\delta)=\Sigma d^{\mu}\tau(\delta)\lambda(\delta)$, $\Sigma\sigma_{\mu}(d)\phi(\delta)=n\sigma_{\mu-1}(n)$, $\Sigma\tau(d^{2\mu})\sigma_v(\delta)=\Sigma d^{v}\tau(\delta)\tau(\delta^{\mu})$, $\Sigma\{\theta(d)\}^{v}\sigma_{\mu}(\delta)=\Sigma d^{\mu}\tau(\delta^{2v})$, $\Sigma\sigma_{\mu}(d)\phi(\delta)=n\sigma_{\mu-1}(n)=\Sigma d\delta^{\mu}$.

The above list, which contains some of the theorems occurring in Dickson, History of the Theory of Numbers, chapters V and X, comprises only some of the many relations which come directly from such a list, and may without difficulty be extended further.

III. Numerical functions in the complex realm, $k(i)$.

1. We shall now consider the generalization of this method to the realm $\sqrt{-1}$ $[k(i)]$, viz., to numbers of the form $a+bi$, where a and b are rational integers. As in the rational realm, R, a function $f(\mu)$, where μ is a number of the form $a+bi$, is called numerical, if for each value of μ, $f(\mu)$ takes a single definite value. We shall first determine, for this realm, the generators of certain numerical functions, suggested by the functions used in R, then from these derive relations between the functions.

2. The method of determining the generator or associate of a given function is similar to that used in R and will be clear from an example. Let, as in R, $\phi(\mu)$, where μ is an integer of $k(i)$, denote the number of integers in a complete residue system, mod μ, which are prime to μ.

Let $\mu=\varepsilon\pi_1^{e_1}\cdot\pi_2^{e_2}\cdot\pi_3^{e_3}\cdots\cdots\pi_r^{e_r}$, be the prime resolution of μ, in which $\varepsilon=\pm1,\pm i$, is a unit of the realm. Then, we have $\phi(\varepsilon)=1$, and $\phi(\mu)=\phi(\pi_1^{e_1})\cdot\phi(\pi_2^{e_2})\cdot\phi(\pi_3^{e_3})\cdots\cdots\phi(\pi_r^{e_r})$.

Consider now the infinite product

$$(1)\quad \Pi_i\left[1+\frac{\phi(\pi_i)}{\pi_i^s}+\frac{\phi(\pi_i^2)}{\pi_i^{2s}}+\cdots\cdots+\frac{\phi(\pi_i^{e_i})}{\pi_i^{e_is}}+\cdots\cdots\right],$$

in which $\pi_i(i=1, 2, 3, \cdots\cdots)$ are all distinct primes of $k(i)$, which can, as in R, be shown absolutely convergent for a suitably chosen $|s|$. In what follows, such an $|s|$ is assumed. The general term in the expansion of (1) is:

$$\frac{\phi(\pi_i^{e_2})\cdot\phi({}_i^{e_2})\cdots\cdots\phi(\pi_i^{e_r})}{\pi_i^{e_is}\cdot\pi_i^{e_2s}\cdots\cdots\pi_i^{e_rs}}=\frac{\phi(\mu)}{\mu^s}.$$

Therefore

$$(2)\qquad \Pi_i\left[1+\frac{\phi(\pi_i)}{\pi_i^s}+\frac{\phi(\pi_i^2)}{\pi_i^{2s}}+\cdots\cdots+\frac{\phi(\pi_i^{e_i})}{\pi_i^{e_is}}+\ \cdots\right]=\sum\frac{\phi(\mu)}{\mu^s}.$$

Since $\phi(\pi^{ei})=n[\pi^{e_i}]-n[\pi^{e_i-1}]$, in which, as usual, $n[\pi]$ denotes the norm of π, (2) becomes, after simplification:

$$\Pi_i\left[\frac{1-1/\pi_i^s}{1-n[\pi_i]/\pi_i^s}\right].$$

We then say, as in R, $\left[\dfrac{1-1/\pi^s}{1-n[\pi]/\pi s}\right]$ is the generator of $\phi(\mu)$, and write

it symbolically $\left[\dfrac{1-1/\pi^s}{1-n[\pi]/\pi^s}\right]\mathcal{F}\ \phi(\mu)$.

3. The whole symbolic notation as given in I, 4. for the rational realm, together with the definitions for units, reciprocals, and compound functions, as given in I, 7., may be carried into the realm of complex numbers, $k(i)$, as follows: replace p by π, where π denotes a prime of $k(i)$; z by ζ, where $\zeta=\pi^{-s}$; and n by μ, where μ denotes any integer of $k(i)$. As before, d, δ denote the conjugate divisors of μ, i. e., $d\delta=\mu$. Hence we may write in conclusion in 2.

$$\left[\frac{1-\zeta}{1-[\pi]\zeta}\right] \cdot F\ \phi(\mu), \text{ and } \frac{1-\zeta}{1-[\pi]\zeta} \sim \phi.$$

Note that in $n[\pi]$, we drop the prefix n, writing it for brevity $[\pi]$, the [ ] denoting "norm of."

4. With this notation in $k(i)$, let us find the infinite product which generates the function $\mu\tau(\mu)$, in which $\tau(\mu)$ denotes the number of divisors of μ.

Let $\mu=\Pi_i\pi_i^{e_i}$, then $\tau(\mu)=\Pi_i(e_i+1)$

and $\mu\tau(\mu)=\Pi_i(e_i+1)\pi e_i$.

Consider the sum:

$$1+\Sigma\pi^e(e+1)\zeta^e$$

which for all values of the integer $e>0$, gives

$$1+2\pi\zeta+3\pi^2\zeta^2+4\pi^3\zeta^3+\cdots\cdots$$
$$=\frac{1}{(1-\pi\zeta)^2}.$$

We may then write:

$$\left[\frac{1}{(1-\pi\zeta)^2}\right] \cdot F\ \mu\tau(\mu), \text{ or } \frac{1}{1-\pi\zeta} \sim (u,\ \tau).$$

5. The list of functions, generators and definitions given in I., 9., and 10. applies to the realm $k(i)$, if in each generator, p is replaced by π, and z by ζ, except in the case of the function $\phi(\mu)$, for which we have R, $\left[\frac{1-z}{1-pz}\right] \cdot F\ \phi(n)$, while in $k(i)$, $\left[\frac{1-\zeta}{1-[\pi]\zeta}\right] \cdot F\ \phi(\mu)$. We may also add for this realm, $\left[\frac{1}{1-[\pi]\rho}\right] \cdot F\ n[\mu]$.

6. From this list, we may derive relations among the functions in $k(i)$, similar to the relations among analogous functions in R. Thus, taking the generators from I, 10, and making the substitutions given in 5, we have:

$$\left[\frac{1-\zeta}{1-[\pi]\zeta}\right] \mathcal{F}\ \phi(\mu) \text{ and } \left[\frac{1}{1-\zeta}\right] \mathcal{F}\ u_0(\mu).$$

Then

$$\left[\frac{1-\zeta}{1-[\pi]\zeta}\cdot\frac{1}{1-\zeta}\right]=\frac{1}{1-[\pi]\zeta}\sim n[\mu].$$

Therefore, we may write $\Sigma\phi(d)=n[\mu]$ in $k(i)$, while in R, we have
$$\Sigma\phi(d)=n.$$

Again, $$\left[\frac{1}{(1-\zeta)(1-\pi\zeta)}\right] \mathcal{F}\ \sigma(\mu) \text{ and } \left[\frac{1}{1-\zeta}\right] \mathcal{F}\ u_0(\mu)$$

$$\left[\frac{1}{(1-\zeta)(1-\pi\zeta)}\cdot\frac{1}{1-\zeta}\right]=\left[\frac{1}{(1-\zeta)^2}\cdot\frac{1}{1-\pi\zeta}\right] \mathcal{F}\ \tau(\mu).u_1(\mu),$$

and we may write $\Sigma\sigma(d)=\Sigma\delta\tau(d)$ or $\sigma u_0=u_1\tau$, which is the same relation as held in R.

7. We thus see that the relations given in II. for integers in R, are true for integers of $k(i)$, with the exception of those involving the ϕ-function, since it's generator in $k(i)$ differs from the one in R. Thus, in R, $\Sigma\phi(\delta)\sigma(d)=n\tau(n)$, while in $k(i)$, we have, taking the generators from I, 10:

$$\left[\frac{1-\zeta}{1-[\pi]\zeta}\cdot\frac{1}{(1-\zeta)(1-\pi\zeta)}\right]=\frac{1}{1-[\pi]\rho}\cdot\frac{1}{1-\pi\rho}\Bigg] \mathcal{F}\ n[\mu].\mu.$$

Hence, in $k(i)$, $\Sigma\phi(\delta)\sigma(d)=\Sigma[d]\delta$.

IV. Extension to the ideals of a realm.

1. Let us consider the generalization of this method for ideals, viz., for systems of integers, $\mu_1, \mu_2, \mu_3, \ldots$, of a realm, infinite in number, such that every linear combination of them, $\lambda_1\mu_1+\lambda_2\mu_2+\ldots$, where $\lambda_1, \lambda_2, \ldots$, are any integers of the realm, is an integer of the system. As usual, we shall denote an ideal by $\mathfrak{m}=(\mu_1, \mu_2, \ldots \mu_r)$ where $\mu_1, \mu_2, \ldots \mu_r$ are the numbers of the realm defining the ideal, but for brevity shall write it simply $\mathfrak{m}$. We shall denote the norm of $\mathfrak{m}$ by $n[\mathfrak{m}]$. Numerical functions will be used in the following sense: a function $f(\mathfrak{m})$, where $\mathfrak{m}$ is any ideal, is called numerical, if for each ideal $\mathfrak{m}$, $f(\mathfrak{m})$ takes a single definite value.

Since we are concerned only with the results of formal addition and subtraction; ideal multiplication and division, and these together are abstractly identical with the four operations of any field, we may

generalize into ideals, the discussions given for R and $k(i)$. We shall first determine, for ideals, the generators of certain numerical functions analogous to the functions considered in the previous realms, and by use of these generators, derive relations among the functions.

2. The method of determining the generators is the same as was used in R and $k(i)$, as will be seen from an example. Let, as usual, $\phi(\mathfrak{m})$, where $\mathfrak{m}$ is any ideal, denote the number of integers of the realm, of a complete residue system, mod $\mathfrak{m}$, which are prime to $\mathfrak{m}$, viz., the number of integers in a reduced residue system, mod $\mathfrak{m}$. Let $\mathfrak{m}=\mathfrak{p}_1^{a_1}\cdot\mathfrak{p}_2^{a_2}\ldots\mathfrak{p}_r^{a_r}$ be the resolution of $\mathfrak{m}$ into prime ideals, $\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_r$, which is possible in one and only one way. We have $\phi(1)=1$, where (1) represents the unit ideal, viz., it contains every integer of the realm, and $\phi(\mathfrak{m})=\phi(\mathfrak{p}_1^{a_1})\cdot\phi(\mathfrak{p}_2^{a_2})\ldots\phi(\mathfrak{p}_r^{a_r})$.

Consider now the infinite product:

$$(1)\qquad \Pi_i\left[1+\frac{\phi(\mathfrak{p}_i)}{n[\mathfrak{p}_i]^s}+\frac{\phi(\mathfrak{p}_i^2)}{n[\mathfrak{p}_i]^{2s}}+\cdots+\frac{\phi(\mathfrak{p}_i^{a_i})}{n[\mathfrak{p}_i]^{a_is}}+\cdots\right]$$

in which $\mathfrak{p}_i(i=1,2,3\ldots)$ are all distinct prime ideals, which can be shown absolutely convergent for a suitably chosen $|s|$, since $n[\mathfrak{p}_i]$ and $\phi(\mathfrak{p}_i)$ are rational integers. Assume such an $|s|$ in what follows.

The general term of the expansion of (1) is:

$$\frac{\phi(\mathfrak{p}_i^{a_1})\phi(\mathfrak{p}_i^{a_2})\ldots\phi(\mathfrak{p}_i^{a_r})}{n[\mathfrak{p}_i]^{a_1s}n[\mathfrak{p}_i]^{a_2s}\ldots n[\mathfrak{p}_i]^{a_rs}}=\frac{\phi(\mathfrak{m})}{n[\mathfrak{m}]^s}.$$

Therefore we have:

$$(2)\qquad \Pi_i\left[1+\frac{\phi(\mathfrak{p}_i)}{n[\mathfrak{p}_i]^s}+\frac{\phi(\mathfrak{p}_i^2)}{n[\mathfrak{p}_i]^{2s}}+\cdots+\frac{\phi(\mathfrak{p}_i^{a_i})}{n[\mathfrak{p}_i]^{a_is}}+\cdots\right]=\sum\frac{\phi(\mathfrak{m})}{n[\mathfrak{m}]^s}.$$

Since $\qquad \phi(\mathfrak{p})=n[\mathfrak{p}]-1,\quad \phi(\mathfrak{p}^a)=n[\mathfrak{p}^a]\left(1-\dfrac{1}{n[\mathfrak{p}]}\right),$

and $n[\mathfrak{p}^s]=\{n[\mathfrak{p}]\}^s$, we have from (2), after obvious reductions

$$\Pi_i\left[\frac{1-\dfrac{1}{n[\mathfrak{p}_i]^s}}{1-\dfrac{n[\mathfrak{p}_i]}{n[\mathfrak{p}_i]^s}}\right]=\sum\frac{\phi(\mathfrak{m})}{n[\mathfrak{m}]^s}.$$

We may then say as in R and $k(i)$:

$$\left(\frac{1-\dfrac{1}{n[\mathfrak{p}]^s}}{1-\dfrac{n[\mathfrak{p}]}{n[\mathfrak{p}]^s}}\right)F\ \phi(\mathfrak{m})\ \text{and}\ \frac{1-\dfrac{1}{n[\mathfrak{p}]^s}}{1-\dfrac{n[\mathfrak{p}]}{n[\mathfrak{p}]^s}}\sim\phi.$$

3. The symbolic notation given in I., 4, for the rational realm, together with the definitions for units, reciprocals, and compound functions, given in I., 7., may be carried over into ideals, as follows: replace p by $\mathfrak{p}$, where $\mathfrak{p}$ represents any prime ideal; let $z=n[\mathfrak{p}]^{-s}$; replace n by $\mathfrak{m}$, where $\mathfrak{m}$ represents any ideal; and let $d\delta$ denote conjugate divisors of $\mathfrak{m}$. i.e. $d\delta=\mathfrak{m}$. We may then write the conclusion in 2. as follows:

$$\left[\frac{1-z}{1-[\mathfrak{p}]z}\right] \cdot\!F \phi\,(\mathfrak{m}) \text{ and } \frac{1-z}{1-[\mathfrak{p}]z} \sim \phi\,.$$

Note that in $n[\mathfrak{p}]$, we drop the suffix n, as in $k(i)$, and write it $[\mathfrak{p}]$, the [ ] denoting "norm of."

4. With this notation, let us find the infinite product which generates the function $\lambda(\mathfrak{m})\tau(\mathfrak{m})$, where $\lambda(\mathfrak{m})$ has the same meaning as in R, viz. $\lambda(\mathfrak{m})=(-1)^{\pi(\mathfrak{m})}$, where $\pi(\mathfrak{m})$ denotes the total number of ideal divisor of $\mathfrak{m}$; and $\tau(\mathfrak{m})$ denotes the number of divisors of $\mathfrak{m}$.

Let $\mathfrak{m}=\Pi_i\,\mathfrak{p}_i^{\alpha_i}$, then $\lambda(\mathfrak{m})=\Pi_i(-1)^{\alpha_i}$, $\tau(\mathfrak{m})=\Pi_i(\alpha_i+1)$ and

$$\lambda(\mathfrak{m})\tau(\mathfrak{m})=\Pi_i(\alpha_i+1)(-1)^{\alpha_i}\,.$$

Consider the sum:

$$1+\Sigma(-1)^{\alpha}(\alpha+1)z^{\alpha}=1+\Sigma(\alpha+1)(-z)^{\alpha}\,,$$

which for all values of the integer $\alpha>0$ gives

$$\frac{1}{(1+z)^2}.$$

We may then write:

$$\left[\frac{1}{(1+z)^2}\right] \cdot\!F\ \lambda(\mathfrak{m})\tau(\mathfrak{m}), \text{ and } \frac{1}{(1+z)^2} \sim (\lambda\tau)\,.$$

5. A list of certain functions, with their generators, analogous to the list given in I, 10 for R, may be given here for ideals.

$\phi(\mathfrak{m})$	$\dfrac{1-z}{1-[\mathfrak{p}]z}$	$\tau(\mathfrak{m})\tau(\mathfrak{m}^{\mu})$	$\dfrac{1+(2\mu-1)z}{(1-z)^3}$
$u_s(\mathfrak{m})$	$\dfrac{1}{1-z}$	$\tau(\mathfrak{m}^{2^{\mu}})$	$\dfrac{1+(2^{\mu}-1)z}{(1-z)^2}$
$n[\mathfrak{m}]$	$\dfrac{1}{1-[\mathfrak{p}]z}$	$\lambda(\mathfrak{m})\tau(\mathfrak{m})$	$\dfrac{1}{(1+z)^2}$
$\mu(\mathfrak{m})$	$1-z$	$\lambda(\mathfrak{m})\tau(\mathfrak{m}^2)$	$\dfrac{1-z^2}{(1+z)^3}$
$\tau(\mathfrak{m})$	$\dfrac{1}{(1-z)^2}$	$\{\mu(\mathfrak{m})\}^2$	$1+z$

$\theta(\mathfrak{m})$	$\dfrac{1+z}{1-z}$	$\{\tau(\mathfrak{m})\}^2$	$\dfrac{1+z}{(1-z)^3}$
$\{\theta(\mathfrak{m})\}^\mu$	$\dfrac{1+(2^\mu-1)z}{1-z}$	$\tau(\mathfrak{m}^2{}_\mu)$	$\dfrac{1+(2\mu-1)z}{(1-z)^2}$
$\lambda(\mathfrak{m})\theta(\mathfrak{m})$	$\dfrac{1-z}{1+z}$	$\lambda(\mathfrak{m})$	$\dfrac{1}{1+z}$
$\tau(\mathfrak{m}^\mu)$	$\dfrac{1+(\mu-1)z}{(1-z)^2}$	$\varepsilon(\mathfrak{m})$	1
$u_1(\mathfrak{m})=\mathfrak{m}$	$\dfrac{1}{1-\mathfrak{p}z}$.		

The function $u_0(\mathfrak{m})$ is the ζ-function for ideals, whose generator $\dfrac{1}{1-z}$ or $\dfrac{1}{1-\dfrac{1}{n[\mathfrak{p}]^s}}$ Landau(¹) has shown to be absolutely convergent. In the generator, $\dfrac{1}{1-\mathfrak{p}z}$, a notation which is in common usage, no meaning of subtraction is to be ascribed to the $-$ sign, since the result of the formal product of such terms, giving the ideal $\mathfrak{m}$, is all we are concerned with.

6. From such a list, we may derive relations among the functions there represented, as follows. Taking the generators from 5, we have:

$$\left[\frac{1+(2\mu-1)z}{(1-z)^3}\right]\mathcal{F}\ \tau(\mathfrak{m})\tau(\mathfrak{m}^\mu),\ \left[\frac{1}{1-z}\right]\mathcal{F}\ u_0(\mathfrak{m}),$$

then
$$\left[\frac{1+(2\mu-1)z}{(1-z)^3}\cdot\frac{1}{1-z}\right]=\left[\frac{1+(2\mu-1)z}{(1-z)^2}\cdot\frac{1}{(1-z)^2}\right].$$

Hence we may write:

$$\Sigma\,\tau(d)\,\tau(d^\mu)\,u_0(\delta)=\Sigma\tau(d^{2\mu})\,\tau(\delta)\,,$$

or since $u_0(\delta)=1$ for all δ,

$$\Sigma\,\tau(d)\tau(d^\mu)=\Sigma\,\tau(d^{2\mu})\,\tau(\delta)\,.$$

Among the relations thus obtained, are:

$$(\mu^2)\lambda=\varepsilon,\quad \mu\tau=u_0,\quad \lambda\theta=u_0,\quad (\lambda\theta)u_0=\lambda,\quad (\lambda\theta)\theta=\varepsilon,$$
$$\phi(\mathfrak{m})=\Sigma n[d]\,\delta\ (^2),$$
$$\theta=u_0(\mu^2)(^3),$$

(¹) Crelle Journal für Mathematik, Vol, 125–126.

(²) A relation given by Landau in Crelle Journal für Mathematik, Vol. 125-126, p. 153.

(³) Landau, Crelle, vol. 125–126, p. 157.

$$\Sigma\theta(d)u_0(\delta)=\tau(\mathfrak{m}^2), \qquad \Sigma\lambda(d)\theta(d)\tau(d^2)=u_0(\mathfrak{m})=1,$$
$$\Sigma\tau(d^{2\mu})=\tau(\mathfrak{m})\tau(\mathfrak{m}^\mu), \qquad \Sigma\lambda(d)\tau(d')=\lambda(\mathfrak{m})\tau(\mathfrak{m}),$$
$$\Sigma\tau(d^2)\lambda(d)=\tau(\mathfrak{m}), \qquad \Sigma\{\theta(d)\}^\mu\tau(\delta)=\Sigma\tau(d^{2^\mu}),$$
$$\Sigma\theta(\delta)\tau(d)\tau(d^\mu)=\Sigma\tau(\delta^2)\tau(d^{2\mu}), \qquad \Sigma\phi(d)=n[\mathfrak{m}],$$
$$\Sigma\tau(d^{2^\mu})\theta(\delta)=\Sigma\{\theta(d)^\mu\tau(\delta^2), \qquad \Sigma\{\theta(d)\}^\mu=\tau(\mathfrak{m}^{2^\mu}),$$
$$\Sigma\theta(d)\tau(\delta)=\{\tau(\mathfrak{m})^2, \qquad \Sigma\{\tau(d)\}^2\lambda(d)\theta(d)=\tau(\mathfrak{m}),$$
$$\Sigma\theta(d)\{\theta(d)\}^\mu=\Sigma\tau(d^{2^\mu})\{u(\delta)\}^2,\ \mu u_0=\varepsilon,\ (\tau)^2\mu=(\mu)^2\tau,\ \theta\,.\,u_0=(\mu)^2\tau.$$

7. The list given in 5 and 6 may easily be extended further, by noting that everything concerning rational integers that involves no other property than that of divisibility or enumeration may be carried over into ideals.

V. Extension to polynomials with respect to a prime ideal modulus (and to a rational prime modulus.)

1. Consider now this method applied to polynomials. The usual definition of divisibility of polynomials with respect to a prime ideal modulus will be used here, viz., a polynomial $f(x)$ is divisible with respect to the modulus $\mathfrak{p}$ ($\mathfrak{p}$ any prime ideal) when there exists a polynomial $\Psi(x)$, such that

$$f(x) \equiv \Psi(x)\phi(x) \bmod \mathfrak{p}$$

in which the coefficients in $f(x)$, $\Psi(x)$, $\phi(x)$, are understood to be integers of any given realm. As usual, $\equiv$ means "identically congruent," viz., if $\phi_1(x) \equiv \Psi_1(x) \bmod \mathfrak{p}$, the corresponding coefficients in $\phi_1(x)$ and $\Psi_1(x)$ are congruent, mod $\mathfrak{p}$. $\Psi(x)$ and $\phi(x)$ are considered divisors of $f(x)$, mod $\mathfrak{p}$. Since the unique factorization theorem with respect to a prime ideal modulus is true for polynomials, we may generalize into polynomials the method of the preceeding sections for those relations which are concerned only with divisibility or enumeration.

2. The entire discussion given in I, 4. and 7. may be carried into polynomials by a change of notation as follows: let p^{-s} be $[P(x)]^{-s}=\xi$; let n be $f(x)$, where $f(x)$ represents any polynomial whose coefficients are integers of the realm; let d, δ be $\phi(x)$, $\Psi(x)$, where $\phi(x)$, $\Psi(x)$ represent conjugate divisors, mod $\mathfrak{p}$ of $f(x)$, viz., $f(x) \equiv \phi(x)\Psi(x) \bmod \mathfrak{p}$.

A corresponding change must also be made in the definitions of the numerical functions used in I, 9., thus $\mu[f(x)]$, Möbius' function, may be

defined here as $=0$ if $f(x)$ is divisible, mod $\mathfrak{p}$, by the square of any polynomial $P(x)$, not a unit; $=+1$ if $f(x)$ is the product, mod $\mathfrak{p}$, of an even number of prime polynomials; $=-1$ if $f(x)$ is the product, mod $\mathfrak{p}$, of an odd number of prime polynomials.

3. The list of generators given in I., 9., and the relations derived from them in II, may be transferred into polynomials by the change in notation given in 2., if we omit those functions which have no meaning here.

4. The method will be clear from an example. Taking the generators from I, 10, and making the changes indicated in 2., we have:

$$\left[\frac{1+(2\mu-1)\xi}{(1-\xi)^3}\right] \boldsymbol{F}\,(\tau[f(x)]\,\tau[f(x)^{\mu}])$$

in which $\tau[f(x)]\tau[f(x)^{\mu}]$ represents the number of divisors, mod $\mathfrak{p}$,

$$\left[\frac{1}{1-\xi}\right] \boldsymbol{F}\, u_0[f(x)]$$

$$\left[\frac{1+(2\mu-1)\xi}{(1-\xi)^3}\cdot\frac{1}{1-\xi}\right]=\left[\frac{1+(2\mu-1)\xi}{(1-\xi)^2}\cdot\frac{1}{(1-\xi)^2}\right] \boldsymbol{F}\,\tau[f(x)^{2\mu}]\,.\,\tau[f(x)].$$

Hence, we may write:

$$\Sigma\tau[\phi(x)]\tau[\phi(x)^{\mu}]u_0[\Psi(x)]=\Sigma\tau[\phi(x)^{2\mu}]\tau[\Psi(x)].$$

Again, we have:

$$\left[\frac{1-\xi}{1+\xi}\right] \boldsymbol{F}\,\lambda[f(x)]\,\theta[f(x)], \qquad \left[\frac{1+\xi}{(1-\xi)^2}\right] \boldsymbol{F}\,\tau[f(x)^2],$$

$$\left[\frac{1-\xi}{1+\xi}\cdot\frac{1+\xi}{(1-\xi)^2}\right]=\left[\frac{1}{1-\xi}\right] \boldsymbol{F}\, u_0[f(x)].$$

Hence, we may write:

$$\Sigma\,\lambda[\phi(x)]\,\theta[\phi(x)]\,\tau[\Psi(x)^2]=u_0[f(x)].$$

5. We shall consider it unnecessary to extend this list of theorems further, as it would be merely a duplication of the preceeding sections.

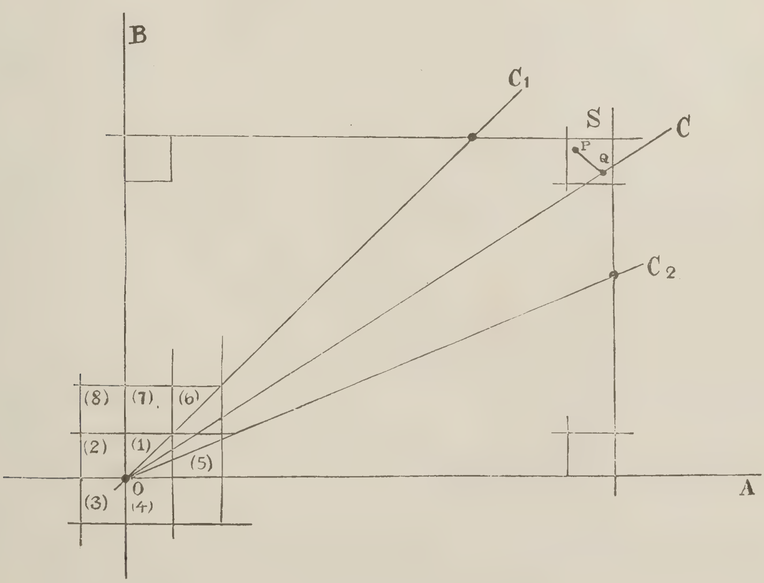
It is evident, that what has been said here for polynomials divisible, mod $\mathfrak{p}$, $\mathfrak{p}$ any prime ideal, would also refer to polynomials divisible, mod p, p a prime of any realm.

On Geometrical Construction by a Ruler of Finite Length and Compasses of Finite Aperture,

by

Motoya Shibayama, Tôkyô.

Steiner proved that all the problems of construction in Elementary Geometry can be solved by means of a ruler only, when a fixed circle is given. But the question: Is this theorem also true, even when the length of a ruler does not exceed some finite magnitude? is worthy to be considered. In the April Meeting of the Mathematical Institute at Sendai, 1918, I have shown that all the problems of construction in Elementary Geometry can be solved by using a ruler of a *finite* length and compasses of a *finite* aperture. Afterwards Mr. Yanagihara has shown that, with certain exceptions, it is sufficient to use a given fixed circle instead of compasses of a finite aperture, and then Mr. Kubota has filled the lack, and has shown that this is true in all cases without



any exception([1]). It may be of some interest to keep my result in record, which precedes the study of these gentlemen.

Let the length of a ruler be l, and the aperture of compasses not exceed the length l. Then, in a circle of radius $l/2$ and on its boundary, all the construction-problems in Euclid's Elements can be applied without hindrance. Hence, it is sufficient for us to ascertain that any two points on a plane can be joined by means of the above described two instruments only.

Let O and P be any two given points Draw any two straight lines OA and OB passing through O and perpendicular to each other, and construct the numbered set of squares, having the sides of length $l/2$, beginning from those by the origin as in the figure. After a finite number of operations, we can determine the square S to which P belongs.

Then, draw the straight line OC_1, bisecting the angle AOB, and next draw OC_2 bisecting the angle AOC_1, and so on. By this process, we can find a straight line OC cutting the square S. Take any point on the part of this line *within* S, and join P and Q. Now, let M_1 be the middle point of PQ, and M_2 be the middle point of M_1Q, and so on. Next, let N_1 be the middle point of OQ, which is easily determined by our restricted instruments. Let N_2 be the middle point of N_1Q, and so on.

Thus we proceed until a pair of points M_k and N_k, both of which belong to S, are attained. Join these two points, and draw the straight line parallel to the join and passing through P. Then, this is the required straight line.

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(1) Yanagihara, On some methods of construction in Elementary Geometry, this Journal, vol. 16, 1919, pp. 48–49; Kubota, Notes in Elementary Geometry, in the same volume of this Journal, pp. 51–52.

On a Sextic,

by

PANDIT OUDH UPADHYAYA, Calcutta, India.

The equation $x^3-5x^2+6x-1=0$, which, writing therein $x+2$ for x gives $x^3+x^2-2x-1=0$, is considered by Hermites Cours d'Analyse, Paris 1873, page 12. This equation suggested to A. Cayley([1]) (as he himself says in the paper referred to) that there are two cubic equations

$$\text{(I) } x^3-3x+1=0\,; \text{ (II) } x^3+x^2-2x-1=0,$$

for each of which roots A, B and C, taken in a proper order are such that

$$2+A=B^2,\ 2+B=C^2 \text{ and } 2+C=A^2,$$

that is to say, these two cubic equations are such that their roots not only satisfy the cubic in question, but also additional three equations.

The object of this paper is to generalize this problem; this paper is divided into two parts. In the first part, it is shown that the roots of the cubic equations

$$x^3-3x+1=0, \text{ and } x^3+x^2-2x-1=0$$

satisfy many other conditions which were not noticed by A. Cayley. In the second part I show that there is an equation of the sixth degree

$$x^6+x^5-5x^4-4x^3+6x^2+3x-1=0,$$

whose roots not only satisfy the sextic and exactly similar six additional equations, but, also many other conditions as well.

I.

The cubic $x^3+x^2-2x-1=0$ has the roots $a+a^6$, a^5+a^2 and a^4+a^3, where a is an imaginary root of $a^7-1=0$. Let the roots be represented by A, C and B respectively, then

$$A=a+a^6,\ C=a^5+a^2 \text{ and } B=a^4+a^3.$$

Now it is evident that

([1]) Messenger of Mathematics, Vol. XXII, 1893, pp. 69–71.

$$A^2=2+C.$$

Similarly the other two equations can be easily proved and these results were given by A. Cayley, though by a different method, but the following equations, satisfied by the roots, were not noticed by A. Cayley:

$$A^3=B+3A,\quad B^3=C+3B,\quad C^3=A+3C.$$

Also $$AB=B+C,\quad AC=A+B.$$

Similarly many other equations can be found which are satisfied by the roots of the cubic $x^3+x-2x-1=0$ and the same is true for the cubic $x^3-3x+1=0$.

II.

The sextic

$$x^6+x^5-5x^4-4x^3+6x^2+3x-1=0$$

is irreducible, in the sense that it cannot be expressed as the product of two factors or three factors of lower degree with rational coefficients; but by adjunction, it can be broken up into the product of linear factors. Let a, 13th root of unity, be adjoined to it, then all its roots can be expressed in terms of a.

Thus

$$x^6+x^5-5x^4-4x^3+6x^2+3x-1=0$$

becomes

$$(x-a-a^{12})(x-a^2-a^{11})(x-a^4-a^9)(x-a^8-a^5)(x-a^3-a^{10})(x-a^6-a^7)=0$$

or

$$x=a+a^{12},\ a^2+a^{11},\ a^4+a^9,\ a^8+a^5,\ a^3+a^{10},\ a^6+a^7.$$

Let the roots be represented by A, B, C, D, E and F. Then

$$A^2=2+B,\quad B^2=2+C,\quad C^2=2+D,$$

$$D^2=2+E,\quad E^2=2+F,\quad F^2=2+A.$$

Thus it is clear that all the roots satisfy six equations, exactly similar to A. Cayley's.

Also $$A^3=E+3A,\quad B^3=F+3B.$$

Similarly other four equations of this type may be determined.

Again $$AB=E+A.$$

Similarly other five equations of this type may be determined.

Again $$A^4=C+4B+6.$$

Similarly other five equations of this type can be determined.

Again $$A^5=D+5E+10A.$$

Similarly other five equations of this type can be easily determined.

In the same way A^3 equated to its value will determine another relations and there will be another five relations of the similar nature.

Also $$ABC=F+A+E+D.$$

Similarly other five equations of this type can be determined.

Thus it is clear that the roots of the equation

$$x^6+x^5-5x^4-4x^3+6x^2+3x-1=0$$

not only satisfy the equation in question but many other equatians as well. Here only 36 such equations have been considered, but the roots of this equation satisfy many other equations, which can be easily determined by the method indicated above.

Cyclotomic Sexe-Section for the Prime 61,

by

PANDIT OUDH UPADHYAYA, Culcutta, India.

The problem of quin-qui-section has recently been solved by W. Burnside in the Proceedings of London Mathematical Society 14 (1915), though those formulæ were first given by L. J. Rogers. The object of this paper is to solve the problem of Sexe-Section for the prime 61. W. Burnside says, in his paper under the heading "Cyclotomic quin-qui-section," "I have carried the $q=7$ so far as to assure myself that it is quite parallel with that of $q=5$. A set of three simultaneous Diophantine relations occur, but they are not sufficient to ensure that the equations expressing the products of the A's form a consistent multiplication table." In view of this fact stated by W. Burnside, it is believed that the problem of Sexe-Section for the prime 61 has not been considered by any previous writer.

Let a be an imaginary root of the equation $x^{61}-1=0$, *i.e.* $a^{61}-1=0$. Then

$$a=1, \text{ or } 1+a+a^2+a^3+a^4+a^5+\cdots+a^{60}=0. \qquad (1)$$

Now dividing all the 60 imaginary roots into six groups according to the following scheme, we can easily find the equation of the sixth degree, whose roots are A, B, C, D, E and F:

$$A=a+a^3+a^9+a^{27}+a^{20}+a^{60}+a^{58}+a^{52}+a^{34}+a^{41},$$
$$B=a^2+a^6+a^{18}+a^{54}+a^{40}+a^{59}+a^{55}+a^{43}+a^7+a^{21},$$
$$C=a^4+a^{12}+a^{36}+a^{47}+a^{19}+a^{57}+a^{49}+a^{25}+a^{14}+a^{42},$$
$$D=a^8+a^{24}+a^{11}+a^{33}+a^{38}+a^{53}+a^{37}+a^{50}+a^{28}+a^{23},$$
$$E=a^{16}+a^{48}+a^{22}+a^5+a^{15}+a^{45}+a^{13}+a^{39}+a^{56}+a^{46},$$
$$F=a^{32}+a^{35}+a^{44}+a^{10}+a^{30}+a^{29}+a^{26}+a^{17}+a^{51}+a^{31}.$$

It is evident that

$$A+B+C+D+E+F=a+a^2+a^3+a^4+a^5+\cdots+a^{60}$$
$$=-1 \text{ by } (1).$$

The number of all combinations of the roots of the sextic taken two at a time is ${}_6C_2$ or 15; and corresponding to each combination,

there will be one multiplication; hence there will be 15 multiplications of the roots taken two at a time; the value of each is as follows:—

(1) $AB = 3a^{60} + 2a^{59} + 3a^{58} + a^{57} + 2a^{56} + 2a^{55} + 2a^{54} + a^{53} + 3a^{52} + a^{51} + a^{50}$
$+ a^{49} + 2a^{48} + a^{47} + 2a^{46} + 2a^{45} + a^{44} + 2a^{43} + a^{42} + 3a^{41} + 2a^{40} + 2a^{39}$
$+ a^{38} + a^{37} + a^{36} + a^{35} + 3a^{34} + a^{33} + a^{32} + a^{31} + a^{30} + a^{29} + a^{28} + 3a^{27}$
$+ a^{26} + a^{25} + a^{24} + a^{23} + 2a^{22} + 2a^{21} + 3a^{20} + a^{19} + 2a^{18} + a^{17} + 2a^{16}$
$+ 2a^{15} + a^{14} + 2a^{13} + a^{12} + a^{11} + a^{10} + 3a^{9} + a^{8} + 2a^{7} + 2a^{6} + 2a^{5} + a^{4}$
$+ 3a^{3} + 2a^{2} + 3a.$

(2) $AC = 2a^{60} + a^{59} + 2a^{58} + 4a^{56} + a^{55} + a^{54} + 2a^{53} + 2a^{52} + a^{51} + 2a^{50} + 4a^{48}$
$+ 4a^{46} + 4a^{45} + a^{44} + a^{43} + 2a^{41} + a^{40} + 4a^{39} + 2a^{38} + 2a^{37} + a^{35} + 2a^{34}$
$+ 2a^{33} + a^{32} + a^{31} + a^{30} + a^{29} + 2a^{28} + 2a^{27} + a^{26} + 2a^{24} + 2a^{23} + 4a^{22}$
$+ a^{21} + 2a^{20} + a^{18} + a^{17} + 4a^{16} + 4a^{15} + 4a^{13} + 2a^{11} + a^{10} + 2a^{9} + 2a^{8}$
$+ a^{7} + a^{6} + 4a^{5} + 2a^{3} + a^{2} + 2a.$

(3) $AD = 2a^{60} + a^{59} + 2a^{58} + 2a^{57} + a^{56} + a^{55} + a^{54} + 2a^{53} + 2a^{52} + 2a^{51} + 2a^{50}$
$+ 2a^{49} + a^{48} + 2a^{47} + a^{46} + a^{45} + 2a^{44} + a^{43} + 2a^{42} + 2a^{41} + a^{40} + a^{39}$
$+ 2a^{38} + 2a^{37} + 2a^{36} + 2a^{35} + 2a^{34} + 2a^{33} + 2a^{32} + 2a^{31} + 2a^{30} + 2a^{29}$
$+ 2a^{28} + 2a^{27} + 2a^{26} + 2a^{25} + 2a^{24} + 2a^{23} + a^{22} + a^{21} + 2a^{20} + 2a^{19} + a^{18}$
$+ 2a^{17} + a^{16} + a^{15} + 2a^{14} + a^{13} + 2a^{12} + 2a^{11} + 2a^{10} + 2a^{9} + 2a^{8} + a^{7} + a^{6}$
$+ a^{5} + 2a^{4} + 2a^{3} + a^{2} + 2a.$

(4) $AE = 2a^{59} + 4a^{57} + 2a^{56} + 2a^{55} + 2a^{54} + a^{53} + a^{51} + a^{50} + 4a^{49} + 2a^{48} + 4a^{47}$
$+ 2a^{46} + 2a^{45} + a^{44} + 2a^{43} + 4a^{42} + 2a^{40} + 2a^{39} + a^{38} + a^{37} + 4a^{36} + a^{35}$
$+ a^{53} + a^{32} + a^{31} + a^{30} + a^{29} + a^{28} + a^{26} + 4a^{25} + a^{24} + a^{23} + 2a^{22} + 2a^{21}$
$+ 4a^{19} + 2a^{18} + a^{17} + 2a^{16} + 2a^{15} + 4a^{14} + 2a^{13} + 4a^{12} + a^{11} + a^{10} + a^{8}$
$+ 2a^{6} + 2a^{7} + 2a^{5} + 4a^{4} + 2a^{2}.$

(5) $AF = 2a^{60} + a^{59} + 2a^{58} + a^{57} + a^{56} + a^{55} + a^{54} + 2a^{53} + 2a^{52} + 3a^{51} + 2a^{50}$
$+ a^{49} + a^{48} + a^{47} + a^{46} + a^{45} + 3a^{44} + a^{43} + a^{42} + 2a^{41} + a^{40} + a^{39} + 2a^{38}$
$+ 2a^{37} + a^{36} + 3a^{35} + 2a^{34} + 2a^{33} + 3a^{32} + 3a^{31} + 3a^{30} + 3a^{29} + 2a^{28}$
$+ 2a^{27} + 3a^{26} + a^{25} + 2a^{24} + 2a^{23} + a^{22} + a^{21} + 2a^{20} + a^{19} + a^{18} + 3a^{17}$
$+ a^{16} + a^{15} + a^{14} + a^{13} + a^{12} + 2a^{11} + 3a^{10} + 2a^{9} + 2a^{8} + a^{7} + a^{6} + a^{5} + a^{4}$
$+ 2a^{3} + a^{2} + 2a.$

(6) $BC = a^{60} + 3a^{59} + a^{58} + 2a^{57} + a^{56} + 3a^{55} + 3a^{54} + a^{53} + a^{52} + 2a^{51} + a^{50}$
$+ 2a^{49} + a^{48} + 2a^{47} + a^{46} + a^{45} + 2a^{44} + 3a^{43} + 2a^{42} + a^{41} + 3a^{40} + a^{39}$
$+ a^{38} + a^{37} + 2a^{36} + 2a^{35} + a^{34} + a^{33} + 2a^{32} + 2a^{31} + 2a^{30} + 2a^{29} + a^{28}$

$$+ a^{27} + 2a^{26} + 2a^{25} + a^{24} + a^{23} + a^{22} + 3a^{21} + a^{20} + 2a^{19} + 3a^{18} + 2a^{17}$$
$$+ a^{16} + a^{15} + 2a^{14} + a^{13} + 2a^{12} + a^{11} + 2a^{10} + a^{9} + a^{8} + 3a^{7} + 3a^{6} + a^{5}$$
$$+ 2a^{4} + a^{3} + 3a^{2} + a.$$

(7) $BD = a^{60} + 2a^{59} + a^{57} + 2a^{56} + 2a^{55} + 2a^{54} + a^{52} + 4a^{51} + a^{49} + 2a^{48} + a^{47}$
$$+ 2a^{46} + 2a^{45} + 4a^{44} + 2a^{43} + a^{42} + a^{41} + 2a^{40} + 2a^{39} + a^{36} + 4a^{35} + a^{34}$$
$$+ 4a^{32} + 4a^{31} + 4a^{30} + 4a^{29} + a^{27} + 4a^{26} + a^{25} + 2a^{22} + 2a^{21} + a^{20} + a^{19}$$
$$+ 2a^{18} + 4a^{17} + 2a^{16} + 2a^{15} + a^{14} + 2a^{13} + a^{12} + 4a^{10} + a^{9} + 2a^{7} + 2a^{6}$$
$$+ 2a^{5} + a^{4} + a^{3} + 2a^{2} + a.$$

(8) $BE = 2a^{60} + 2a^{59} + 2a^{58} + a^{57} + 2a^{56} + 2a^{55} + 2a^{54} + 2a^{53} + 2a^{52} + a^{51} + 2a^{50}$
$$+ a^{49} + 2a^{48} + a^{47} + 2a^{46} + 2a^{45} + a^{44} + 2a^{43} + a^{42} + 2a^{41} + 2a^{40} + 2a^{39}$$
$$+ 2a^{38} + 2a^{37} + a^{36} + a^{35} + 2a^{34} + 2a^{33} + a^{32} + a^{31} + a^{30} + a^{29} + 2a^{28}$$
$$+ 2a^{27} + a^{26} + a^{25} + 2a^{24} + 2a^{23} + 2a^{22} + 2a^{21} + 2a^{20} + a^{19} + 2a^{18} + a^{17}$$
$$+ 2a^{16} + 2a^{15} + a^{14} + 2a^{13} + a^{12} + 2a^{11} + a^{10} + 2a^{9} + 2a^{8} + 2a^{7} + 2a^{6}$$
$$+ 2a^{5} + a^{4} + 2a^{3} + 2a^{2} + 2a.$$

(9) $BF = a^{60} + a^{58} + 2a^{57} + a^{56} + 4a^{53} + a^{52} + 2a^{51} + 4a^{50} + 2a^{49} + a^{48} + 2a^{47}$
$$+ a^{46} + a^{45} + 2a^{44} + 2a^{42} + a^{41} + a^{39} + 4a^{38} + 4a^{37} + 2a^{36} + 2a^{35} + a^{34}$$
$$+ 4a^{33} + 2a^{32} + 2a^{31} + 2a^{30} + 2a^{29} + 4a^{28} + a^{27} + 2a^{26} + 2a^{25} + 4a^{24}$$
$$+ 4a^{23} + a^{22} + a^{20} + 2a^{19} + 2a^{17} + a^{16} + a^{15} + 2a^{14} + a^{13} + 2a^{12} + 4a^{11}$$
$$+ 2a^{10} + a^{9} + 4a^{8} + a^{5} + 2a^{4} + a^{3} + a.$$

(10) $CD = 2a^{60} + a^{59} + 2a^{58} + 3a^{57} + a^{56} + a^{55} + a^{54} + 2a^{53} + 2a^{52} + a^{51} + 2a^{50}$
$$+ 3a^{49} + a^{48} + 3a^{47} + a^{46} + a^{45} + a^{44} + a^{43} + 3a^{42} + 2a^{41} + a^{40} + a^{39} + 2a^{38}$$
$$+ 2a^{37} + 3a^{36} + a^{35} + 2a^{34} + 2a^{33} + a^{32} + a^{31} + a^{30} + a^{29} + 2a^{28} + 2a^{27}$$
$$+ a^{26} + 3a^{25} + 2a^{24} + 2a^{23} + a^{22} + a^{21} + 2a^{20} + 3a^{19} + a^{18} + a^{17} + a^{16} + a^{15}$$
$$+ 3a^{14} + a^{13} + 3a^{12} + 2a^{11} + a^{10} + 2a^{9} + 2a^{8} + a^{7} + a^{6} + a^{5} + 3a^{4} + 2a^{3}$$
$$+ a^{2} + 2a.$$

(11) $CE = 4a^{60} + a^{59} + 4a^{58} + 2a^{57} + a^{55} + {}^{54} + a^{53} + 4a^{52} + 2a^{51} + a^{50} + 2a^{49}$
$$+ 2a^{47} + 2a^{44} + a^{43} + 2a^{42} + 4a^{41} + a^{40} + a^{38} + a^{37} + 2a^{36} + 2a^{35} + 4a^{34}$$
$$+ a^{33} + 2a^{32} + 2a^{31} + 2a^{30} + 2a^{29} + a^{28} + 4a^{27} + 2a^{26} + 2a^{25} + a^{24} + a^{23}$$
$$+ a^{21} + 4a^{20} + 2a^{19} + a^{18} + 2a^{17} + 2a^{14} + 2a^{12} + a^{11} + 2a^{10} + 4a^{9} + a^{8}$$
$$+ a^{7} + a^{6} + 2a^{4} + 4a^{3} + a^{2} + 4a.$$

(12) $CF = a^{60} + 2a^{59} + a^{58} + 2a^{57} + 2a^{56} + 2a^{55} + 2a^{54} + a^{53} + a^{52} + 2a^{51} + a^{50}$
$$+ 2a^{49} + 2a^{48} + 2a^{47} + 2a^{46} + 2a^{45} + 2a^{44} + 2a^{43} + 2a^{42} + a^{41} + 2a^{40}$$

$+2a^{39}+a^{38}+a^{37}+2a^{36}+2a^{35}+a^{34}+a^{33}+2a^{32}+2a^{31}+2a^{30}+2a^{29}$
$+a^{28}+a^{27}+2a^{26}+2a^{25}+a^{24}+a^{23}+2a^{22}+2a^{21}+a^{20}+2a^{19}+2a^{18}$
$+2a^{17}+2a^{16}+2a^{15}+2a^{14}+2a^{13}+2a^{12}+a^{11}+2a^{10}+a^{9}+a^{8}+2a^{7}$
$+2a^{6}+2a^{5}+2a^{4}+a^{3}+2a^{2}+a.$

(13) $DE=a^{60}+2a^{59}+a^{58}+a^{57}+2a^{56}+2a^{55}+2a^{54}+3a^{53}+a^{52}+a^{51}+3a^{50}$
$+a^{49}+2a^{48}+a^{47}+2a^{46}+2a^{45}+a^{44}+2a^{43}+a^{42}+a^{41}+2a^{40}+2a^{39}$
$+3a^{38}+3a^{37}+a^{36}+a^{35}+a^{34}+3a^{33}+a^{32}+a^{31}+a^{30}+a^{29}+3a^{28}+a^{27}$
$+a^{26}+a^{25}+3a^{24}+3a^{23}+2a^{22}+2a^{21}+a^{20}+a^{19}+2a^{18}+a^{17}+2a^{16}$
$+2a^{15}+a^{14}+2a^{13}+a^{12}+3a^{11}+a^{10}+a^{9}+3a^{8}+2a^{7}+2a^{6}+2a^{5}+a^{4}$
$+a^{3}+2a^{2}+a.$

(14) $DF=2a^{60}+4a^{59}+2a^{58}+a^{57}+a^{56}+4a^{55}+4a^{54}+2a^{53}+2a^{52}+2a^{50}+a^{49}$
$+a^{48}+a^{47}+a^{46}+a^{45}+4a^{43}+a^{42}+2a^{41}+4a^{40}+a^{39}+2a^{38}+2a^{37}$
$+a^{36}+2a^{34}+2a^{33}+2a^{28}+2a^{27}+a^{25}+2a^{24}+2a^{23}+a^{22}+4a^{21}+2a^{20}$
$+a^{19}+4a^{18}+a^{16}+a^{15}+a^{14}+a^{13}+a^{12}+2a^{11}+2a^{9}+2a^{8}+4a^{7}+4a^{6}$
$+a^{5}+a^{4}+2a^{3}+4a^{2}+2a.$

(15) $EF=a^{60}+a^{59}+a^{58}+2a^{57}+3a^{56}+a^{55}+a^{54}+a^{53}+2a^{51}+a^{52}+a^{50}+2a^{49}$
$+3a^{48}+2a^{47}+3a^{46}+3a^{45}+2a^{44}+a^{43}+2a^{42}+a^{41}+a^{40}+3a^{39}+a^{38}$
$+a^{37}+2a^{36}+2a^{35}+a^{34}+a^{33}+2a^{32}+2a^{31}+2a^{30}+2a^{29}+a^{28}+a^{27}$
$+2a^{26}+2a^{25}+a^{24}+a^{23}+3a^{22}+a^{21}+a^{20}+2a^{19}+a^{18}+2a^{17}+3a^{16}$
$+3a^{15}+2a^{14}+3a^{13}+2a^{12}+a^{11}+2a^{10}+a^{9}+a^{8}+a^{7}+a^{6}+3a^{5}+2a^{4}$
$+a^{3}+a^{2}+a.$

Adding all these together, we get

$$\Sigma AB=25(a+a^2+a^3+a^4+\cdots\cdot a^{60})=-25.$$

Similarly we can calculate ΣABC; but in order to save space only two of the multlplications will be given.

(1) $ABC=10+19a+16a^2+19a^3+13a^4+19a^5+16a^6+16a^7+16a^8$
$+19a^9+16a^{10}+16a^{11}+13a^{12}+19a^{13}+13a^{14}+19a^{15}+19a^{16}$
$+16a^{17}+16a^{18}+13a^{19}+19a^{20}+16a^{21}+19a^{22}+16a^{23}+16a^{24}$
$+13a^{25}+16a^{26}+19a^{27}+16a^{28}+16a^{29}+16a^{30}+16a^{31}+16a^{32}$
$+16a^{33}+19a^{34}+16a^{35}+13a^{36}+16a^{37}+16a^{38}+19a^{39}+16a^{40}$
$+19a^{41}+13a^{42}+16a^{43}+16a^{44}+19a^{45}+19a^{46}+13a^{47}+19a^{48}$
$+13a^{49}+16a^{50}+16a^{51}+19a^{52}+16a^{53}+16a^{54}+16a^{55}+19a^{56}$
$+13a^{57}+19a^{58}+16a^{59}+19a^{60}.$

$$\begin{aligned}(2)\ DEF = {} & 10 + 16a + 19a^2 + 16a^3 + 16a^4 + 16a^5 + 19a^6 + 19a^7 + 19a^8 \\ & + 16a^9 + 13a^{10} + 19a^{11} + 16a^{12} + 16a^{13} + 16a^{14} + 16a^{15} + 16a^{16} \\ & + 13a^{17} + 19a^{18} + 16a^{19} + 16a^{20} + 19a^{21} + 16a^{22} + 19a^{23} + 19a^{24} \\ & + 16a^{25} + 13a^{26} + 16a^{27} + 19a^{28} + 13a^{29} + 13a^{30} + 13a^{31} + 13a^{32} \\ & + 19a^{33} + 16a^{34} + 13a^{35} + 16a^{36} + 19a^{37} + 19a^{38} + 16a^{39} + 19a^{40} \\ & + 16a^{41} + 16a^{42} + 19a^{43} + 13a^{44} + 16a^{45} + 16a^{46} + 16a^{47} + 16a^{48} \\ & + 16a^{49} + 19a^{50} + 13a^{51} + 16a^{52} + 19a^{53} + 19a^{54} + 19a^{55} + 16a^{56} \\ & + 16a^{57} + 16a^{58} + 19a^{59} + 16a^{60}.\end{aligned}$$

Calculating all the terms of the similar types (their number is ${}_6C_3$, i. e. 20) and adding, we get

$$\Sigma ABC = 320 + 328(a + a^2 + a^3 + a^4 + \cdots + a^{60}) = -8.$$

Similarly

$$\Sigma ABCD = 2580 + 2457(a + a^2 + a^3 + a^4 + \cdots + a^{60}) = 123.$$

In the same way

$$\Sigma ABCDE = 10020 + 9833(a + a^2 + a^3 + a^4 + \cdots + a^{60}) = 187,$$

$$ABCDEF = 16420 + 16393(a + a^2 + a^3 + a^4 + \cdots + a^{60}) = 27.$$

Therefore the sextic is:

$$x^6 + x^5 - 25x^4 + 8x^3 + 123x^2 - 187x + 27 = 0.$$

Every root of this equation may be expressed as a rational integral function of any one assigned root and therefore this is an Abelian equation which can be solved by radicals.

Second paper on Tautochronous Motion,

by

PANDIT OUDH UPADHYAYA, Calcuta, India

In a previous paper published in this Journal, vol. (1922) pp. 32–36, I considered the general problem of tautochronous motion and gave the general rule for every curve whose expression for the arc can be integrated in a finite number of terms. I showed in that paper that there are one-fold infinity of solutions for the problem in question. If, however, the forces are taken to be at right angles to each other and one of them is supposed to be zero, problem becomes very easy and is solved at once.

The object of this paper is to apply the general rule to some well known particular cases and it is believed that these have not been considered by any previous writer with the help of this method.

Case I. Circle.

A particle is made to describe a circle $x^2+y^2=a^2$ under the action of attraction varying as $\frac{1}{y}\sin^{-1}\frac{x}{a}$ from the axis of x and perpendicular to it. Prove that the curve is a tautochrone for this law of force.

In the circle

$$S=a\sin^{-1}\frac{x}{a},\quad \frac{ds}{dx}=\frac{a}{y}.$$

According to the question

$$\frac{dx^2}{dt^2}=-\frac{\mu}{y}\sin^{-1}\frac{x}{a},\text{ and }\frac{dy^2}{dt^2}=0.$$

Therefore

$$\frac{ds^2}{dt^2}=\frac{dx^2}{df^2}\frac{dx}{ds}+\frac{dy^2}{dt^2}\frac{dy}{ds}=-\mu\frac{s}{a^2},$$

or

$$\frac{ds^2}{dt^2}+u'S=0.$$

Therefore this is a tautochronous motion.

Case II. A cubic curve $y=x^3/3$.

A particle is made to describe a curve $y=x^3/3$ under the action of an attraction which varies as $(1+x^4)^2$ from the axis of x and is perpendicular to it. Then the curve is a tautochrone for the law of force.

Case III. $y=\log \sin x$.

A particle is made to describe a logarithmic curve whose equation is $y=\log\sin x$, under the action of a force which varies directly as $\log\left(\tan \dfrac{x}{2}\right)\times \operatorname{cosec} x$ and perpendicular to it. Then the curve is a tautochrone for this law of force.

Case IV. Logarithmic curve $y=b\,e^{\frac{x}{c}}$.

A particle is made to describe a logarithmic curve under the attraction, which varies as $\dfrac{\sqrt{c^2+y^2}}{y}\left\{c\log\dfrac{y}{c+\sqrt{c^2+y^2}}+\sqrt{c^2+y^2}\right\}$ from the axis of y and is perpendicular to it. Then the curve is a tautochrone for this law of force.

On the Integral $\int\limits_0^\infty x^{-m} \sin^n dx$, with an Appendix on its Application to a Theory of Approximation of a Function,

by

TSURUICHI HAYASHI, Sendai.

In the Nieuw Archief voor Wiskunde, tweede reeks, deel 13, 1920, p. 324, I have dealt with the evaluation of such integrals in which n is a positive integer and m is a positive number, but not necessarily an integer, under the supposition that

$$\text{when } n \text{ is odd}: \qquad 0<m<n+1,$$

$$\text{when } n \text{ is even}: \qquad 1<m<n+1,$$

in order that the integral is convergent. Prof. J. C. Kluyver, Editor of the Nieuw Archief, has kindly informed me that another method of evaluation leads to a result of a rather simple form in the special case that m is an integer and that $n-m$ is even.

His results are:

Putting

$$\frac{\pi}{2}\cdot\frac{1\cdot3\cdot5\cdot7\cdots(2k-1)}{2\cdot4\cdot6\cdot8\cdots2k}=A(k),$$

$$\int\limits_0^\infty\frac{\sin^{2k+1}x}{x}dx=A(k),$$

$$\int\limits_0^\infty\frac{\sin^{2k+1}x}{x^3}dx=A(k-1)-\frac{1}{2}A(k),$$

$$\int\limits_0^\infty\frac{\sin^{2k+1}x}{x^5}dx=A(k-2)-\frac{5}{6}A(k-1)+\frac{1}{24}A(k),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$\int\limits_0^\infty\frac{\sin^{2k}x}{x^2}dx=A(k-1),$$

$$\int_0^\infty \frac{\sin^{2k}x}{x^4}dx = A(k-2) - \frac{2}{3}A(k-1),$$

$$\int_0^\infty \frac{\sin^{2k}x}{x^6}dx = A(k-3) - A(k-2) + \frac{2}{15}A(k-1),$$

. . . .

Now I will give a general rule to obtain the numerical coefficients of the functions A's.

First of all, let me evaluate

$$\int_0^\infty \frac{\sin^{2k+1}x}{x}dx \qquad \text{and} \qquad \int_0^\infty \frac{\sin^{2k}x}{x^2}dx.$$

By the well known formula,

$$2^{2k}(-1)^k \sin^{2k+1}x = \sum_{r=0}^{k}(-1)^r\binom{2k+1}{r}\sin(2k+1-2r)x.$$

Hence immediately

$$2^{2k}(-1)^k\int_0^\infty \frac{\sin^{2k+1}x}{x}dx = \sum_{r=0}^{k}(-1)^r\binom{2k+1}{r}\frac{\pi}{2}.$$

By the also well known addition-theorem of binomial coefficients,

$$\binom{n+m}{k} = \binom{n}{k}\binom{m}{0} + \binom{n}{k-1}\binom{m}{1} + \binom{n}{k-2}\binom{m}{2} + \cdots\cdots + \binom{n}{0}\binom{m}{k}.$$

Putting $m=-1$,

$$(-1)^k\binom{n-1}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots\cdots + (-1)^k\binom{n}{k}.$$

Hence

$$\sum_{r=0}^{k}(-1)^r\binom{2k+1}{r} = (-1)^k\binom{2k}{k}.$$

Therefore

$$\int_0^\infty \frac{\sin^{2k+1}x}{x}dx = \frac{\pi}{2}\cdot\frac{1\cdot3\cdot5\cdot7\cdots\cdots(2k-1)}{2\cdot4\cdot6\cdot8\cdots\cdots 2k} = A(k).$$

Next, by integration by parts

$$\int_0^\infty \frac{\sin^{2k}x}{x^2}dx = -\int_0^\infty \sin^{2k}x\,d(x^{-1})$$

$$=\left|-\frac{\sin^{2k}x}{x^2}\right|_0^\infty+\int_0^\infty\frac{D\sin^{2k}x}{x}\,dx$$

$$=\int_0^\infty\frac{D\sin^{2k}x}{x}\,dx,$$

where D stands for $\dfrac{d}{dx}$. Now by the well known formula

$$2^{2k-1}(-1)^k\sin^{2k}x=\sum_{r=0}^{k}(-1)^r\binom{2k}{r}\cos 2\,(k-r)x\,;$$

whence

$$2^{2k-1}(-1)^k D\sin^{2k}x=-\sum_{r=0}^{k}(-1)^r\binom{2k}{r}\cdot 2(k-r)\sin 2\,(k-r)x$$

$$=-\sum_{r=0}^{k-1}(-1)^r\binom{2k}{r}\cdot 2(k-r)\sin 2(k-r)x.$$

Hence

$$2^{2k-1}(-1)^k\int_0^\infty\frac{D\sin^{2k}x}{x}\,dx=-\sum_{r=0}^{k-1}(-1)^r\binom{2k}{r}\cdot 2\,(k-r)\cdot\frac{\pi}{2}$$

$$=\left[-2k\sum_{r=0}^{k-1}(-1)^r\binom{2k}{r}+2\sum_{r=0}^{k-1}(-1)^r\binom{2k}{r}r\right]\cdot\frac{\pi}{2}.$$

But

$$\sum_{r=0}^{k-1}(-1)^r\binom{2k}{r}=(-1)^{k-1}\binom{2k-1}{k-1},$$

and

$$\sum_{r=0}^{k-1}(-1)^r\binom{2k}{r}r=2k\sum_{r=1}^{k-1}(-1)^r\binom{2k-1}{r-1}$$

$$=-2k\sum_{r=0}^{k-2}(-1)^r\binom{2k-1}{r}$$

$$=(-1)^{k-1}2k\binom{2k-2}{k-2}.$$

Therefore

$$2^k\int_0^\infty\frac{\sin^{2k}x}{x^2}\,dx=\left\{2k\binom{2k-1}{k-1}-4k\binom{2k-2}{k-2}\right\}\frac{\pi}{2}$$

$$=2k\frac{(2k-2)\,!}{(k-1)!\,k!}\cdot\frac{\pi}{2},$$

and therefore

$$2^{k-1}\int_0^\infty \frac{\sin^{2k}x}{x^2}dx = \frac{1\cdot3\cdot5\cdot7\cdots(2k-3)}{(k-1)!}\ \frac{\pi}{2},$$

i. e. $$\int_0^\infty \frac{\sin^{2k}x}{x^2}dx = \frac{\pi}{2}\cdot\frac{1\cdot3\cdot5\cdot7\cdots(2k-3)}{2\cdot4\cdot6\cdot8\cdots(2k-2)} = A(k-1).$$

Now I proceed to evaluate the integrals

$$\int_0^\infty \frac{\sin^{2k+1}x}{x^{2\mu+1}}dx \qquad \text{and} \qquad \int_0^\infty \frac{\sin^{2k}x}{x^{2\mu}}dx.$$

By a formula in my preceding paper above cited and already published

$$\int_0^\infty \frac{\sin^n x}{x^m}dx = \frac{1}{(m-1)(m-2)\cdots(m-s)}\int_0^\infty \frac{D^s\sin^n x}{x^{m-s}}dx.$$

where $s<m$.

Firstly, put

$$n=2k+1,\ m=2\mu+1 \text{ and } s=2\mu.$$

Then

$$\int_0^\infty \frac{\sin^{2k+1}x}{x^{2\mu+1}}dx = \frac{1}{(2\mu)!}\int_0^\infty \frac{D^{2\mu}\ \sin^{2k+1}x}{x}dx.$$

Secondly, put

$$n=2k,\ m=2\mu \text{ and } s=2\mu-2.$$

Then

$$\int_0^\infty \frac{\sin^{2k}x}{x^{2\mu}}dx = \frac{1}{(2\mu-1)!}\int_0^\infty \frac{D^{2\mu-2}\sin^{2k}x}{x^2}dx.$$

Therefore if we have a method to express

$$D^{2\mu}\sin^{2k+1}x$$

in odd powers of $\sin x$, and to express

$$D^{2\mu-2}\sin^{2k}x$$

in even powers of $\sin x$, we can express the said integrals in terms of the functions A's.

Now

$$D^2 \sin^n x = n(n-1) \sin^{n-2} x - n^2 \sin^n x,$$

and

$$D^4 \sin^n x = n(n-1)(n-2)(n-3) \sin^{n-4} x$$
$$- n(n-1)\{n^2 + (n-2)^2\} \sin^{n-2} x + n^4 \sin^n x.$$

Hence assume that

$$D^{2\mu} \sin^n x = N_{\mu,0} \sin^n x + N_{\mu,1} \sin^{n-2} x + \cdots + N_{\mu,\mu} \sin^{n-2\mu} x,$$

in which the coefficients $N_{\mu,r}$ are functions of n, μ and r, and differentiate the equation twice with respect to x. Then by comparison of coefficients we get the recurrence-formula

$$\begin{aligned}
N_{\mu+1,0} &= -n^2 N_{\mu,0}, \\
N_{\mu+1,1} &= -(n-2)^2 N_{\mu,1} + n(n-1) N_{\mu,0}, \\
N_{\mu+1,2} &= -(n-4)^2 N_{\mu,2} + (n-2)(n-3) N_{\mu,1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
N_{\mu+1,\mu} &= -(n-2\mu)^2 N_{\mu,\mu} + (n-2\mu+2)(n-2\mu+1) N_{\mu,\mu-1}, \\
N_{\mu+1,\mu+1} &= (n-2\mu)(n-2\mu-1) N_{\mu,\mu}.
\end{aligned}$$

By the first of these formulae,

$$N_{\mu,0} = (-n^2)^\mu.$$

Then we can get successively $N_{\mu,1}$, $N_{\mu,2}, \cdots$ from the other formulae, and get lastly

$$N_{\mu,\mu} = n(n-1)(n-2) \cdots (n-2\mu-2).$$

This having been effected, we have the final results

$$\int_0^\infty \frac{\sin^{2k+1} x}{x^{2\mu+1}} dx = \frac{1}{(2\mu)!} \{ N_{\mu,0} A(k) + N_{\mu,1} A(k-1) + \cdots + N_{\mu,\mu} A(k-\mu) \},$$

where $N_{\mu,r}$ are functions of $2k+1$, μ and r; and

$$\int_0^\infty \frac{\sin^{2k} x}{x^{2\mu}} dx = \frac{1}{(2\mu-1)!} \{ N_{\mu-1,0} A(k-1) + N_{\mu-1,1} A(k-2) + \cdots$$
$$+ N_{\mu-1,\mu-1} A(k-\mu) \},$$

where $N_{\mu,r}$ are functions of $2k$, μ and r.

February 1921.

Appendix.

Prof. de la Vallée-Poussin proves in his Leçons sur l'approximation des fonctions d'une variable réelle, 1919, p. 45, the theorem that if a function satisfies a Lipschitz condition it can be represented by a trigonometric sum of order n with a maximum error not exceeding $A\omega(1/n)$, where A is an absolute constant, and $\omega(\delta)$ is the so-called modulus of continuity of the function, and he calculates a rough approximation of the constant A, as to be

$$A=1+\frac{6}{\pi}\int_0^\infty \frac{\sin^4 x}{t^3}\,dt<1+\frac{6}{\pi}<3.$$

In the Bulletin of the American Mathematical Society, vol. 28, 1922, pp. 59–61, we find a review of the Leçons by Porof. Dunham Jackson, who writes on p. 61 that the relation

$$\int_0^\infty \frac{\sin^4 t}{t^3}\,dt=\log 2$$

is useful for the exact calculation of the constant and adds in the footnote on the same page that "I do not remember seeing a proof of this relation in print; I am personally indebted for various demonstrations of it to Messrs. Gronwall, Landau, M. Riesz and I. Schur. But the relation is printed already in the famous work of Bierens de Haan, Nouvelles tables d'intégrales définies, 1867, Table 156. My demonstration of a more general relation is already printed in the Nieuw Archief voor Wiskunde, tweede reeks, deel 13, p. 330. Formula [D] gives the value of the integral

$$\int_0^\infty \frac{\sin^n t}{t^m}\,dt,$$

when n is an even positive integer and m is an odd positive integer. Particularizing this formula by putting $n=4$, $m=3$ we easily find the said relation.

May 1922. T. H.

Projective and Correlative Applicabilities of Two Surfaces,

by

Tsurusaburo Takasu, Sendai.

Recently Prof. Fubini in Torino published a very elegant essay entitled *Applicabilità proiettiva di due superficie*(1) and Cartan(2) made a further study on the same topic. In this paper, following Fubini's example, though his condition seems to be of much limited a nature for applications, I will found up *a condition for correlative applicability* and then *another condition for projective applicability of two surfaces*(3). The concept of the *reduced dual relative total curvature* will thereby play an important rôle(4). I have neglected the notion of *geometrical applicability* of Fubini(5).

§1. Two Fundamental Cubic Forms.

1. Notation.

$$X\equiv y_1,\ Y\equiv y_2,\ Z\equiv y_3,\ W\equiv y_4\equiv -xX-yY-zZ,\ X^2+Y^2+Z^2\equiv 1.$$

$$y_i=y_i(u,v),\qquad i=1,2,3,4.$$

$$\mathfrak{p}_i\equiv\frac{\partial y_i}{\partial u},\quad \mathfrak{q}_i\equiv\frac{\partial y_i}{\partial v},\quad \mathfrak{r}_i\equiv\frac{\partial^2 y_i}{\partial u^2},\quad \mathfrak{s}_i\equiv\frac{\partial^2 y_i}{\partial u\partial v},\quad \mathfrak{t}_i\equiv\frac{\partial^2 y_i}{\partial v^2},$$

$$\mathfrak{a}_i\equiv\frac{\partial^3 y_i}{\partial u^3},\quad \mathfrak{b}_i\equiv\frac{\partial^3 y_i}{\partial u^2\partial v},\quad \mathfrak{c}_i\equiv\frac{\partial^3 y_i}{\partial u\partial v^2},\quad \mathfrak{d}_i\equiv\frac{\partial^3 y_i}{\partial v^3}.$$

$$\varphi_2\equiv(y\mathfrak{p}\mathfrak{q}d^2y)\equiv(y,\mathfrak{p},\mathfrak{q},\mathfrak{r}du^2+2\mathfrak{s}du\,dv+\mathfrak{t}dv^2),$$

$$g_3\equiv(y\mathfrak{p}\mathfrak{q}d^3y)\equiv(y,\mathfrak{p},\mathfrak{q},3[\mathfrak{r}du\,d^2u+\mathfrak{s}(du\,d^2v+d^2u\,dv)+\mathfrak{t}du\,d^2v])$$
$$+(y,\mathfrak{p},\mathfrak{q},[\mathfrak{a}du^3+3\mathfrak{b}du^2dv+3\mathfrak{c}du\,dv^2+\mathfrak{d}dv^3]).$$

$$Y_i=Y_i(u,v),\qquad i=1,2,3,4.$$

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(1) Guido Fubini, Applicabilità proiettiva di due superficie, Rend. d. Circ. Mat. d. Palermo, 41 (1916).

(2) Cartan, Sur la déformation projective des surfaces, Annales scientifiques de l'École normale supérieure. 3e série, Tome 37, N° 9 (1920), p. 259.

(3) The noneuclidean case may be treated almost similarly.

(4) Takasu, Differential Geometry of Surfaces in Euclidean Plane-space. The Science Reports of the Tôhoku Imperial University, vol. II, No. 5 (1922), p. 411.

(5) See Art. 8.

$$\mathfrak{P}_i \equiv \frac{\partial Y_i}{\partial u}, \text{ etc.,} \quad \mathfrak{R}_i \equiv \frac{\partial^2 Y_i}{du^2}, \text{ etc.,} \quad \mathfrak{A}_i \equiv \frac{\partial^3 Y_i}{du^3}, \text{ etc..}$$

$$\Psi_2 \equiv (Y \mathfrak{P} \mathfrak{Q}\, d^2 Y), \qquad G_3 \equiv (Y \mathfrak{P} \mathfrak{Q}\, d^3 Y), \text{ etc..}$$

Fubini introduced the following notation:

$$x_1 \equiv x, \quad x_2 \equiv y, \quad x_3 \equiv z, \quad x_4 \equiv 1.$$

$$x_i \equiv x_i(u, v), \qquad i = 1, 2, 3, 4.$$

$$p_i \equiv \frac{\partial x_i}{\partial u}, \quad q_i \equiv \frac{\partial x_i}{\partial v}, \quad r_i \equiv \frac{\partial^2 x_i}{du^2}, \quad s_i \equiv \frac{\partial^2 x_i}{du dv}, \quad t_i \equiv \frac{\partial^2 x_i}{\partial v^2},$$

$$\alpha_i \equiv \frac{\partial^3 x_i}{\partial u^3}, \quad \beta_i \equiv \frac{\partial^3 x_i}{\partial u^2 \partial v}, \quad \gamma_i \equiv \frac{\partial^3 x_i}{\partial u \partial v^2}, \quad \delta_i \equiv \frac{\partial^3 x_i}{\partial v^3}.$$

$$(xpqr) \equiv (pqr), \text{ etc..}$$

$$\varphi_2 \equiv (xpq\, d^2 x),$$
$$\equiv (x, p, q,\ r du^2 + 2s du dv + t dv^2),$$
$$f_3 \equiv (xpq\, d^3 x),$$
$$\equiv (x, p, q,\ 3[r du d^2 u + s(du d^2 v + d^2 u dv) + t dv d^2 v])$$
$$+ (x, p, q, [\alpha du^3 + 3\beta du^2 dv + 3\gamma du dv^2 + \delta dv^3]).$$

$$X_i \equiv X_i(u, v).$$

$$P_i \equiv \frac{\partial X_i}{\partial u}, \text{ etc.,} \quad R_i \equiv \frac{\partial^2 X_i}{\partial u^2}, \text{ etc.,} \quad A_i \equiv \frac{\partial^3 X_i}{\partial u^3}, \text{ etc.}$$

$$\Phi_2 \equiv (XPQ d^2 X)$$

$$F_3 \equiv (XPQ d^3 X), \text{ etc..}$$

2. A correlation:

$$y_i = \sum_k a_{ik} x_k.$$

$$(a_{11}\, a_{22}\, a_{33}\, a_{44}) \equiv D.$$

(i) $$(y\mathfrak{p}\mathfrak{q}\mathfrak{r}) = D.\,(xpqr),$$

(ii) $$\psi_2 = D.\,\varphi_2,$$

(iii) $$g_3 = D.\,f_3.$$

$$d\varphi_2 = (x, dp, q,\ r du^2 + 2s du dv + t dv^2)$$
$$+ (x, p, dq,\ r du^2 + 2s du dv + t dv^2)$$
$$+ (x, p, q,\ 2[r du d^2 u + s(du d^2 v + d^2 v du) + t dv d^2 v])$$
$$+ (x, p, q,\ \alpha du^3 + 3\beta du^2 dv + 3\gamma du dv^2 + \delta dv^3).$$

If we put

$$\varphi_3 \equiv 3\,d\varphi_2 - 2f_3, \qquad \psi_3 \equiv 3d\psi_2 - 2g_3,$$

we have

(iv) $$\psi_3 = D.\,\varphi_3.$$

A collineation:

$$x_i = \sum_k b_{ik}\,X_k,$$

i.e. $$y_i = \sum_k b_{ki}\,Y_k.$$

$$(b_{11}\,b_{22}\,b_{33}\,b_{44}) \equiv \Delta.$$

(v) $$(xpqr) = \Delta.(XPQR), \quad (y\mathfrak{p}\mathfrak{q}\mathfrak{r}) = \Delta.(Y\mathfrak{P}\mathfrak{Q}\mathfrak{R}),$$

(vi) $$\psi_2 = \Delta.\,\Psi_2, \qquad \varphi_2 = \Delta.\,\Phi_2,$$

(vii) $$\psi_3 = \Delta.\,\Psi_3, \qquad \varphi_3 = \Delta.\,\Phi_3.$$

3. If $L = L(u, v)$ be an arbitrary function of u, v, and put $L\varphi_2$, Lf_3, $L\psi_2$, Lg_3 for $\varphi_2, f_3, \psi_2, g_3$ respectively, then we have

$$3d(L\varphi_2) - 2Lf_3 \qquad \text{and} \qquad 3d(L.\,\psi_2) - 2L.\,g_3$$

in place of

$$\varphi_3 \qquad \text{and} \qquad \psi_3$$

respectively.

N. B. Both $\varphi_2 = 0$ and $\psi_2 = 0$ are the differential equations of asymptotic lines. Thus by collineations as well as by correlations, the asymptotic lines are preserved.

4. If $\rho = \rho(u, v)$ and if we put

$$[\varphi_2] \equiv [\varphi_2]_{x//\rho x}, \text{ etc.}, \quad [\psi_2] \equiv [\psi_2]_{y//\rho y}, \text{ etc.},$$

(i) $[\varphi_2] = \rho^4.\,\varphi_2$,	(i) $[\psi_2] = \rho^4.\,\psi_2$,
(ii) $[d\varphi_2] = 4\rho^3\,d\rho\varphi_2 + \rho^4\,d\varphi_2$,	(ii) $[d\psi_2] = 4\rho^3\,d\rho\psi_2 + \rho^4\,d\psi_2$,
(iii) $[f_3] = \rho^4.\,f_3 + 3\rho^3\,\varphi_2\,d\rho$,	(iii) $[g_3] = \rho^4\,g_3 + 3\rho^3\psi_2\,d\rho$,
(iv) $[\varphi_3] = \rho^4\,\varphi_3 + 6(\rho^3 d\rho)\,\varphi_2$.	(iv) $[\psi_3] = \rho^4\,\psi_3 + 6(\rho^3 d\rho)\,\psi_2$.

N. B. Asymptotic lines are preserved by the substitution $y//\rho y$ or $x//\rho x$.

5. $$u = u(u', v'), \qquad v = v(u', v').$$

$$\frac{\partial x_i}{\partial u'} = \frac{\partial x_i}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial x_i}{\partial v}\frac{\partial v}{\partial u'}, \quad \frac{\partial x_i}{\partial v'} = \frac{\partial x_i}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial x_i}{\partial v}\frac{\partial v}{\partial v'}, \text{ etc.}.$$

(i) $\varphi_2' = \dfrac{\partial(u, v)}{\partial(u', v')}\varphi_2,$

(ii) $f_3' = \dfrac{\partial(u, v)}{\partial(u', v')}f_3,$

(iii) $\varphi_3' = \dfrac{\partial(u, v)}{\partial(u', v')}\varphi_3 + 3\varphi_2 d\dfrac{\partial(u, v)}{\partial(u', v')}.$

(i) $\psi_2' = \dfrac{\partial(u, v)}{\partial(u', v')}\psi_2,$

(ii) $g_3' = \dfrac{\partial(u, v)}{\partial(u', v')}g_3,$

(iii) $\psi_3' = \dfrac{\partial(u, v)}{\partial(u', v')}\psi_3 + 3\psi_2 d\dfrac{\partial(u, v)}{\partial(u', v')}.$

Proof. $\varphi_2' = (xp'q'd^2x),$

$$= \left(x,\ \frac{\partial x}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial u'},\ \frac{\partial x}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial v'},\ d^2x\right),$$

$$= (x\,p\,q\,d^2x)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\partial u}{\partial u'} & \dfrac{\partial v}{\partial u'} & 0 \\ 0 & \dfrac{\partial u}{\partial v'} & \dfrac{\partial v}{\partial v'} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$= \frac{\partial(u, v)}{\partial(u', v')}\varphi_2.$$

Similarly $$\psi_2' = \frac{\partial(u, v)}{\partial(u', v')}\psi_2.$$

(ii) and (iii) may be proved similarly.

N.B. Asymptotic lines are preserved by the transformation $u = u(u', v')$, $v = v(u', v')$.

6.

(I) $[\varphi_3] = 3[d\varphi_2] - 2[f_3],$
$= \rho^4\varphi_3 + 6\rho^3 d\rho\varphi_2.$

(I) $[\psi_3] = 3[d\psi_2] - 2[g_3],$
$= \rho^4\psi_3 + 6\rho^3 d\rho\psi_2.$

Put $$du = \xi, \qquad dv = \eta.$$

Problem. If we fix the point (plane) (u, v) on the surface (x) $((y))$, the righthand side of (I), which contains the indeterminate parts

$$\rho,\ \frac{\partial\rho}{\partial u},\ \frac{\partial\rho}{\partial v}$$

represents a linear system of binary cubic forms. It is required to individualize one of them by a purely projective method, considering (ξ, η)

as homogeneous coordinates on a one-dimensional primitive form([1]).

Solution. Let the cubic forms whose Hessian is

(1) $$\varphi_2 \equiv b_{11}\xi^2 + 2b_{12}\xi\eta + b_{22}\eta^2$$

be

$$F \equiv a\xi^3 + b\xi^2\eta + c\xi\eta^2 + d\eta^3,$$

whose Hessian is

(2) $$\begin{vmatrix} 3a\xi + b\eta & b\xi + c\eta \\ b\xi + c\eta & 3d\eta + c\xi \end{vmatrix}.$$

Identifying (1) with (2), we have

$$\left.\begin{aligned} 3ac - b^2 &= b_{11}, \\ 9ad - bc &= 2b_{12}, \\ 3bd - c^2 &= b_{22} \end{aligned}\right\}, \qquad \begin{aligned} a &= \frac{b_{11}+b^2}{3c}, \\ b &= \frac{b_{22}+c^2}{3b}. \end{aligned}$$

Substituting in F and multiplying by $3abc$, we have

$$b(b_{11}+b^2)\xi^3 + 3b^2c\,\xi^2\eta + 3bc^2\,\xi\eta^2 + c(b_{22}+c^2)\eta^3.$$

Put $b \equiv \mu$, $c \equiv \lambda$, then F becomes

$$\mu(b_{11}+\mu^2)\xi^3 + 3\mu^2\lambda\xi^2\eta + 3\mu\lambda^2\xi\eta^2 + \lambda(b_{22}+\lambda^2)\eta^3.$$

Now

$$9ad - bc = (b_{11}+b^2)(b_{22}+c^2)/bc - bc = 2b_{12}.$$

Therefore

$$b_{11}b_{22} + b^2 b_{22} + c^2 b_{11} = 2b_{12}bc,$$

i.e.

$$b_{11}b_{22} + \mu^2 b_{22} + \lambda^2 b_{11} = 2b_{12}\lambda\mu,$$

whence

$$(b_{11}+\mu^2) = \lambda(2b_{12}\mu - \lambda b_{11})/b_{22},$$

$$(b_{22}+\mu^2) = \mu(2b_{12}\lambda - \mu b_{22})/b_{11}.$$

Therefore F is

$$\lambda\mu(2b_{12}\mu - \lambda b_{11})b_{22}^{-1}\xi^3 + 3\mu^2\lambda\xi^2\eta + 3\mu\lambda^2\xi\eta^2 + \lambda\mu(2b_{12}\lambda - \mu b_{22})b_{11}^{-1}\eta^3$$
$$= \lambda\mu b_{11}^{-1}b_{22}^{-1}[b_{11}(2b_{12}\mu - \lambda b_{11})\xi^3 + 3(\mu\xi^2\eta + \lambda\xi\eta^2)b_{11}b_{22} + b_{22}(2b_{12}\lambda - \mu b_{22})\eta^3].$$

Therefore

$$\lambda(-b_{11}^2\xi^3 + 3b_{11}b_{22}\xi\eta^2 + 2b_{12}b_{22}\eta^3) + \mu(2b_{11}b_{22}\xi^3 + 3b_{11}b_{22}\xi^2\eta - b_{22}^2\eta^3) = 0.$$

([1]) This problem has already been solved by Fubini for the lefthand side. Here we have to treat the righthand side. The processes are parallel.

Now the system of forms (I) is of the type

$$(\mathrm{I}') \qquad h_1(b_{111}\xi^3+3b_{112}\xi^2\eta+3b_{122}\xi\eta^2+b_{222}\eta^3) \\ +(h_2\xi+h_3\eta)(b_{11}\xi^2+2b_{12}\xi\eta+b_{22}\eta^2),$$

where h's are variable parameters.

In order that one of the forms (I′) may belong to this system, must

$$b_{111}+\bar{h}\,b_{11}=-\lambda b_{11}^2+2\mu b_{11}b_{12},$$
$$b_{222}+\bar{k}\,b_{22}=-\mu b_{22}^2+2\lambda b_{22}b_{11},$$
$$3b_{112}+2\bar{h}\,b_{12}+\bar{k}\,b_{11}=3\mu b_{11}b_{22},$$
$$3b_{221}+\bar{h}\,b_{22}+2\bar{k}\,b_{12}=3\lambda b_{11}b_{22},$$

where $\qquad \bar{h}\equiv h_2/h_1, \qquad \bar{k}\equiv h_3/h_1.$

Substituting the values of λ, μ from the last two into the first two, we have

$$\bar{h}=\frac{3}{4}\cdot\frac{2b_{12}b_{112}-b_{11}b_{122}-b_{111}b_{22}}{b_{11}b_{22}-b_{12}^2},$$
$$\bar{k}=\frac{3}{4}\cdot\frac{2b_{12}b_{122}-b_{22}b_{112}-b_{222}b_{11}}{b_{11}b_{22}-b_{12}^2},$$

which determine our forms

$$\varphi_3+(\bar{h}\xi+\bar{k}\eta)\varphi_2 \qquad | \qquad \psi_3+(\bar{h}\xi+\bar{k}\eta)\psi_2$$

completely except a factor.

7. Now let us transform the cubic forms just obtained.

$$\psi_2=(y\ \mathfrak{p}\ \mathfrak{q}\ d^2y),$$

$$d^2y_i=\frac{\partial^2 y_i}{\partial u^2}du^2+2\frac{\partial^2 y_i}{\partial u\partial v}dudv+\frac{\partial^2 y_i}{\partial v^2}dv^2, \text{ etc.}$$

Therefore

$$\psi_2=\left(y\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\frac{\partial^2 y}{\partial u^2}\right)du^2+2\left(y\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\frac{\partial^2 y}{\partial u\partial v}\right)dudv \\ +\left(y\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\frac{\partial^2 y}{\partial v^2}\right)dv^2.$$

Therefore

$$\psi_2/\Delta=Ddu^2+2D'dudv+D''dv^2=\Sigma xd^2y.$$

$$\varphi_3=(y\ \mathfrak{p}\ \mathfrak{q}\ d^3y),$$
$$=\Sigma\left(y_2\frac{\partial y_3}{\partial u}\frac{\partial y_4}{\partial v}\right)d^3y_1,$$

$$x_1 \bar{\Delta} = \left(y_2 \frac{\partial y_3}{\partial u} \frac{\partial y_4}{\partial v} \right), \text{ etc..}$$

Therefore

$$g_3 = \bar{\Delta} \, \Sigma x \, d^3 y.$$

Therefore

$$\begin{aligned}
3d(\psi_2/\bar{\Delta}) - 2g_3/\bar{\Delta} &= 3d(D du^2 + 2D' du dv + D'' dv^2) - 2\Sigma x d^3 y, \\
&= \left(3\frac{\partial D}{\partial u} - 2\Sigma x \frac{\partial^3 y}{\partial u^3}\right) du^3 \\
&\quad + 3\left(\frac{\partial D}{\partial v} + 2\frac{\partial D'}{\partial u} - 2\Sigma x \frac{\partial^3 y}{\partial u^2 \partial v}\right) du^2 dv \\
&\quad + 3\left(\frac{\partial D''}{\partial u} + 2\frac{\partial D'}{\partial v} - 2\Sigma x \frac{\partial^3 y}{\partial u \partial v^2}\right) du\, dv^2 \\
&\quad + \left(3\frac{\partial D''}{\partial v} - 2\Sigma x \frac{\partial^3 y}{\partial v^3}\right) dv^3 \\
&\quad + 6\left(D - \Sigma x \frac{\partial^2 y}{\partial u^2}\right) du\, d^2 u \\
&\quad + 6\left(D' - \Sigma x \frac{\partial^2 y}{\partial u \partial v}\right)(d^2 u\, dv + du\, d^2 v) \\
&\quad + 6\left(D'' - \Sigma x \frac{\partial^2 y}{\partial v^2}\right) dv\, d^2 v,
\end{aligned}$$

where

$$D = \Sigma x \frac{\partial^2 y}{\partial u^2}, \quad D' = \Sigma x \frac{\partial^2 y}{\partial u \partial v}, \quad D'' = \Sigma x \frac{\partial^2 y}{\partial v^2}.$$

Put

$$\begin{aligned}
\bar{D}_u &\equiv 3\frac{\partial D}{\partial u} - 2\Sigma x \frac{\partial^3 y}{\partial u^3}, \\
\bar{D}_v &\equiv \frac{\partial D}{\partial v} + 2\frac{\partial D'}{\partial u} - 2\Sigma x \frac{\partial^3 y}{\partial u^2 \partial v}, \\
\bar{D}_u'' &\equiv \frac{\partial D''}{\partial u} + 2\frac{\partial D'}{\partial v} - 2\Sigma x \frac{\partial^3 y}{\partial u \partial v^2}, \\
\bar{D}_v'' &\equiv 3\frac{\partial D''}{\partial v} - 2\Sigma x \frac{\partial^3 y}{\partial v^3},
\end{aligned}$$

then we have

$$\begin{aligned}
3d(\psi_2/\bar{\Delta}) - 2g_3/\bar{\Delta} &\equiv \bar{D}_u du^3 + 3\bar{D}_v du^2 dv + 3\bar{D}_v'' du^2 dv + 3\bar{D}_u'' du dv^2 + \bar{D}_v'' dv^3, \\
&\equiv \psi_3.
\end{aligned}$$

Now let us prove that

$$\left\{\begin{aligned}
\overline{D}_u &= \frac{\partial D}{\partial u} - 2\left\{ {11 \atop 1} \right\}' D - 2\left\{ {11 \atop 2} \right\}' D', \\
\overline{D}_v &= \frac{\partial D}{\partial u} - \left\{ {21 \atop 1} \right\}' D - \left\{ {11 \atop 2} \right\}' D' - \left\{ {21 \atop 2} \right\}' D - \left\{ {11 \atop 2} \right\}' D'', \\
\overline{D}_u'' &= \frac{\partial D''}{\partial u} - 2\left\{ {12 \atop 1} \right\}' D - 2\left\{ {12 \atop 2} \right\}' D'', \\
\overline{D}_v'' &= \frac{\partial D''}{\partial v} - 2\left\{ {22 \atop 1} \right\}' D' - 2\left\{ {22 \atop 2} \right\}' D'',
\end{aligned}\right.$$

$$D = \sum_{i=1}^{i=4} x_i \frac{\partial^2 y_i}{\partial u^2}.$$

$$\frac{\partial D}{\partial u} = \sum_{i=1}^{i=3} \frac{\partial x_i}{\partial u} \frac{\partial^2 y_i}{\partial u^2} + \sum_{i=1}^{i=4} x_i \frac{\partial^3 y_i}{\partial u^3}.$$

$$\frac{\partial^2 y_i}{\partial u^2} = \left\{ {11 \atop 1} \right\}' \frac{\partial y_i}{\partial u} + 2\left\{ {11 \atop 2} \right\}' \frac{\partial y_i}{\partial v} - \mathcal{E} y_i, \quad i = 1, 2, 3.$$

Therefore

$$\sum_{i=1}^{i=3} \frac{\partial x_i}{\partial u} \frac{\partial^2 y_i}{\partial u^2} = \left\{ {11 \atop 1} \right\}' \sum_{i=1}^{i=3} \frac{\partial y_i}{\partial u} \frac{\partial x_i}{\partial u} + \left\{ {11 \atop 2} \right\}' \sum_{i=1}^{i=3} \frac{\partial y_i}{\partial v} \frac{\partial x_i}{\partial u} - \mathcal{E} \sum_{i=1}^{i=4} y_i \frac{\partial x_i}{\partial u},$$

$$= \left\{ {11 \atop 1} \right\}' D - \left\{ {11 \atop 2} \right\}' D'.$$

Therefore $$\sum_{i=1}^{i=4} x_i \frac{\partial^3 y_i}{\partial u^3} = \frac{\partial D}{\partial u} + \left\{ {11 \atop 1} \right\}' D + \left\{ {11 \atop 2} \right\}' D'.$$

Therefore $$\overline{D}_u \equiv 3\frac{\partial D}{\partial u} - 2\sum_{i=1}^{i=4} x_i \frac{\partial^3 y_i}{\partial u^3},$$

$$= \frac{\partial D}{\partial u} - 2\left\{ {11 \atop 1} \right\}' D - 2\left\{ {11 \atop 2} \right\}' D'.$$

Again $$\frac{\partial D}{\partial v} = \sum_{i=1}^{i=3} \frac{\partial x_i}{\partial v} \frac{\partial^2 y_i}{\partial u^2} + \sum_{i=1}^{i=4} x_i \frac{\partial^3 y_i}{\partial u^2 \partial v},$$

$$D' = \sum_{i=1}^{i=4} x_i \frac{\partial^2 y_i}{\partial u \partial v},$$

$$\frac{\partial D'}{\partial u} = \sum_{i=1}^{i=3} \frac{\partial x_i}{\partial u} \frac{\partial^2 y_i}{\partial u dv} + \sum_{i=1}^{i=4} x_i \frac{\partial^3 y_i}{\partial u^2 \partial v}.$$

$$\frac{\partial^2 y_i}{\partial u \partial v} = \left\{ {12 \atop 1} \right\}' \frac{\partial y_i}{\partial u} + \left\{ {12 \atop 2} \right\}' \frac{\partial y_i}{\partial v} - \mathcal{F} y_i.$$

$$\sum_{i=1}^{i=3} \frac{\partial x_i}{\partial u} \frac{\partial^2 y_i}{\partial u \partial v} = \left\{ {12 \atop 1} \right\}' \sum_{i=1}^{i=3} \frac{\partial y_i}{\partial u} \frac{\partial x_i}{\partial u} + \left\{ {12 \atop 2} \right\}' \sum_{i=1}^{i=3} \frac{\partial y_i}{\partial v} \frac{\partial x_i}{\partial u} - \mathcal{F} \sum_{i=1}^{i=3} y_i \frac{\partial y_i}{\partial u},$$

$$= -\left\{ {12 \atop 1} \right\}' D - \left\{ {12 \atop 2} \right\}' D'.$$

Therefore

$$\frac{\partial D'}{\partial u} = -\left\{ {12 \atop 1} \right\}' D - \left\{ {12 \atop 2} \right\}' D + \sum_{i=1}^{i=4} x_i \frac{\partial^3 y_i}{\partial u^2 \partial v}.$$

$$\bar{D}_v \equiv \frac{\partial D}{\partial v} + 2\frac{\partial D'}{\partial u} - 2\sum_{i=1}^{i=4} x_i \frac{\partial^3 y_i}{\partial u^2 \partial v},$$

$$= \frac{\partial D}{\partial v} - 2\left\{ {12 \atop 1} \right\}' D - 2\left\{ {12 \atop 2} \right\}' D'.$$

Similarly for the other two.

Now our form is

$$\psi_3 + (h\xi + k\eta)\psi_2/\bar{\Delta} \equiv (\bar{D}_u du^3 + 3\bar{D}_v du^2 dv + 3\bar{D}_u'' du dv^2 + \bar{D}_v'' dv^3)$$
$$+ (h du + k dv)(D du^2 + 2D' du dv + D'' dv^2),$$
$$\equiv (b_{111}\xi^3 + 3b_{112}\xi^2\eta + 3b_{122}\xi\eta^2 + b_{222}\eta^3)$$
$$+ (\bar{h}\xi + \bar{k}\eta)(b_{11}\xi^2 + 2b_{12}\xi\eta + b_{22}\eta^2),$$

where

$$\bar{h} = \frac{3}{4}\frac{2b_{12}b_{112} - b_{11}b_{122} - b_{111}b_{22}}{b_{11}b_{22} - b_{12}^2},$$

$$= 3[2D'\bar{D}_u' - D\bar{D}_u'' - D''\bar{D}_u].$$

$$[2D'\bar{D}_u' - D\bar{D}_u'' - D''\bar{D}_u]$$

$$= 2D'\left[\frac{\partial D'}{\partial u} - \left\{ {12 \atop 1} \right\}' D - \left\{ {12 \atop 2} \right\}' D' - \left\{ {11 \atop 1} \right\}' D' - \left\{ {11 \atop 2} \right\}' D''\right]$$

$$- D\left[\frac{\partial D''}{\partial u} - 2\left\{ {12 \atop 1} \right\}' D' - 2\left\{ {12 \atop 2} \right\}' D''\right]$$

$$- D''\left[\frac{\partial D}{\partial u} - 2\left\{ {11 \atop 1} \right\}' D - 2\left\{ {11 \atop 2} \right\}' D'\right]$$

$$= -\frac{\partial}{\partial u}(DD'' - D'^2) + 2\left(\left\{ {11 \atop 1} \right\}' + \left\{ {12 \atop 2} \right\}'\right)(DD'' - D'^2),$$

$$= -\left[\frac{\partial \log(DD'' - D'^2)}{\partial u} - \frac{\partial \log(\mathcal{E}\mathcal{G} - \mathcal{F}^2)}{\partial u}\right](DD'' - D'^2),$$

since

$$\left\{ {11 \atop 1} \right\}' + \left\{ {12 \atop 2} \right\}' \equiv \frac{\partial \log \bar{\Delta}}{\partial u},$$

$$= -\frac{\partial \log \dfrac{DD'' - D'^2}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}}{\partial u}(DD'' - D'^2).$$

Hence

$$\bar{h} = -\frac{3}{4}\frac{\partial \log \bar{\mathfrak{K}}}{\partial u},$$

where $\bar{\mathfrak{K}}$ the *reduced dual total relative curvature*(1). Similarly, we have

$$\bar{k} = -\frac{3}{4}\frac{\partial \log \bar{\mathfrak{K}}}{\partial v}.$$

Fubini has shown(2) that

$$\begin{aligned}
&\varphi_3 + (h\xi + k\eta)\varphi_2/\varDelta \\
&\quad \equiv (D_u\, du^3 + 3D_v\, du^2\, dv + 3D_u''\, du\, dv^2 + D_v''\, dv^3) \\
&\qquad + (h\xi + k\eta)(D du^2 + 2D'\, du\, dv + D''\, dv^2),
\end{aligned}$$

where

$$D_u \equiv \frac{\partial D}{\partial u} - 2\left\{ {11 \atop 1} \right\} D - 2\left\{ {11 \atop 2} \right\} D',$$

$$D_v \equiv \frac{\partial D}{\partial u} - \left\{ {21 \atop 2} \right\} D - \left\{ {11 \atop 1} \right\} D' - \left\{ {21 \atop 2} \right\} D' - \left\{ {11 \atop 2} \right\} D'',$$

$$D_u'' = \frac{\partial D''}{\partial u} - 2\left\{ {12 \atop 1} \right\} D' - 2\left\{ {12 \atop 2} \right\} D'',$$

$$D_v'' = \frac{\partial D''}{\partial v} - 2\left\{ {22 \atop 1} \right\} D' - 2\left\{ {22 \atop 2} \right\} D'',$$

$$h = -\frac{3}{4}\frac{\partial \log K}{\partial u},$$

$$k = -\frac{3}{4}\frac{\partial \log K}{\partial v},$$

K being the total curvature.

§ 2. Vicinities of the First or Second Order.

8. Fubini distinguished *analytical vicinity* from *geometrical vicinity* and so *analytical contact* from *geometrical contact*. But as Cartan has indicated(3), it seems to be of rather artificial a nature.

(1) See Takasu, Differential Geometry of Surfaces in Euclidean Plane-space, I. The Science Reports of the Tôhoku Imperial University, I. Series, vol. II. (1922), p. 411.

(2) Loc. cit.

(3) Loc. cit.

Since the following theorems hold, of which the former was proved by Fubini(1) and the latter is provable similarly, such a distinction will no doubt be unnecessary at least for the present purpose—

1°, 2°. If two surfaces are geometrically applicable to the second order by means of a collineation (correlation), they are also analytically applicable.

9. *Definition.* S and Σ are two surfaces, whose points (planes) are in one-to-one correspondence in such a way that the homologous points (planes) have the same coordinates u, v. We say that *S and Σ have a common vicinity of n-th order* ($n=1, 2$) *at the point A (on the plane α)*, when $A\,(\alpha)$ belongs to both surfaces and the differentials of orders not higher than n of the coordinates (x) (or (y)) are at A (on α) equal for the two surfaces.

10. First let us study about two curves γ and Γ in one-to-one correspondence defined by the following equations

$$\gamma : x_i = x_i(t), \quad \Gamma : X_i = X_i(t), \quad i=1, 2, 3, 4. \qquad \Bigg| \qquad \gamma : y_i = y_i(t), \quad \Gamma : Y_i = Y_i(t), \quad i=1, 2, 3, 4.$$

In order that the two curves may have a common corresponding point A (plane α), we must have for a certain value of t

(I) $\quad \bar{x}_i = \bar{X}_i, \ i=1, 2, 3, \qquad$ | (I) $\quad \bar{y}_i = \bar{Y}_i, \ i=1, 2, 3,$

where

$$\bar{x}_i = x_i/x_4, \text{ etc.} \qquad \Big| \qquad \bar{y}_i = y_i/y_4, \text{ etc.}.$$

In order that the two curves may have a contact of the first order at A (on α), we must have

(II) $\quad \bar{x}_i' = \bar{X}_i', \ i=1, 2, 3. \qquad$ | (II) $\quad \bar{y}_i' = \bar{Y}_i', \ i=1, 2, 3.$

There will take place a contact of the second order, when at A (on α) we have

(III) $\quad \bar{x}_i'' = \bar{X}_i'', \ i=1, 2, 3. \qquad$ | (III) $\quad \bar{y}_i'' = \bar{Y}_i'', \ i=1, 2, 3.$

11. Consider now two surfaces

$$S : \bar{x}_i = \bar{x}_i(u, v), \quad \Sigma : \bar{X}_i = \bar{X}_i(u, v), \quad i=1, 2, 3, \qquad \Bigg| \qquad S : \bar{y}_i = \bar{y}_i(u, v), \quad \Sigma : \bar{Y}_i = \bar{Y}_i(u, v), \quad i=1, 2, 3,$$

(1) Loc. cit.

having a common corresponding point A (tangent plane α), such that for the corresponding values of u, v, the relation

(I) $\bar{x}_i = \bar{X}_i, \ i=1, 2, 3$ | (I) $\bar{y}_i = \bar{Y}_i, \ i=1, 2, 3$

holds.

In order that each pair of corresponding curves

$$u=u(t), \qquad v=v(t)$$

of S, Σ may have contact of the first order, must

(II) $\bar{x}_i' = \bar{X}_i', \ i=1, 2, 3,$ | (II) $\bar{y}_i' = \bar{Y}_i', \ i=1, 2, 3,$

that is, in order that S, Σ may have a common element, must

(III)

$$\bar{x}_{iu}'u' + \bar{x}_{iv}'v' \equiv \bar{X}_{iu}'u' + \bar{X}_{iv}'v', \quad i=1, 2, 3$$

at A,

(III)

$$\bar{y}_{iu}'u' + \bar{y}_{iv}'v' \equiv \bar{Y}_{iu}'u + \bar{Y}_{iv}'v', \quad i=1, 2, 3$$

on α,

i.e.

$$\text{(IV)} \quad \begin{cases} \bar{x}_{iu}' = \bar{X}_{iu}', \\ \bar{x}_{iv}' = \bar{X}_{iv}', \ i=1, 2, 3. \end{cases} \qquad \text{(IV)} \quad \begin{cases} \bar{y}_{iu}' = \bar{Y}_{iu}', \\ \bar{y}_{iv}' = \bar{Y}_{iv}', \ i=1, 2, 3. \end{cases}$$

In order that each pair of corresponding curves of S, Σ may have a contact of the second order, must

$$\text{(V)} \begin{cases} \dfrac{\partial^2 \bar{x}_i}{\partial u^2} = \dfrac{\partial^2 \bar{X}_i}{\partial u^2}, \\ \dfrac{\partial^2 \bar{x}_i}{\partial u \partial v} = \dfrac{\partial^2 \bar{X}_i}{\partial u \partial v}, \quad i=1, 2, 3. \\ \dfrac{\partial^2 \bar{x}_i}{\partial v^2} = \dfrac{\partial^2 \bar{X}_i}{\partial v^2}. \end{cases} \qquad \text{(V)} \begin{cases} \dfrac{\partial^2 \bar{y}_i}{\partial u^2} = \dfrac{\partial^2 \bar{Y}_i}{\partial u^2}, \\ \dfrac{\partial^2 \bar{y}_i}{\partial u \partial v} = \dfrac{\partial^2 \bar{Y}_i}{\partial u \partial v}, \quad i=1, 2, 3. \\ \dfrac{\partial^2 \bar{y}_i}{\partial v^2} = \dfrac{\partial^2 \bar{Y}_i}{\partial v^2}. \end{cases}$$

12.

(VI)

$$\begin{cases} x = \lambda X_i, \\ p = \lambda(P_i + mX_i), \\ q_i = \lambda(Q_i + nX_i), \\ r_i = \lambda(R_i + 2mP_i + aX_i), \\ s_i = \lambda(S_i + mQ_i + nP_i + bX_i), \\ t = \lambda(T_i + 2nQ_i + cX_i), \end{cases}$$

(VI)

$$\begin{cases} y_i = \lambda Y_i, \\ \mathfrak{p}_i = \lambda(\mathfrak{P}_i + mY_i), \\ \mathfrak{q}_i = \lambda(\mathfrak{Q}_i + nY_i), \\ \mathfrak{r}_i = \lambda(\mathfrak{R}_i + 2m\mathfrak{P}_i + aY_i), \\ \mathfrak{s}_i = \lambda(\mathfrak{S}_i + m\mathfrak{Q}_i + n\mathfrak{P}_i + bY), \\ \mathfrak{t}_i = \lambda(\mathfrak{T}_i + 2n\mathfrak{Q}_i + cY_i), \end{cases}$$

where

$$\lambda \equiv x_4/X_4,$$
$$m \equiv p_4/x_4 - P_4/X_4,$$
$$n \equiv q_4/x_4 - Q_4/X_4,$$
$$a \equiv r_4/x_4 - R\,/X_4 - 2mP_4/X_4,$$
$$b \equiv s_4/x_4 - S_4/X_4 - nP_4/X_4 - mQ_4/X_4,$$
$$c \equiv t_4/x_4 - T_4/X_4 - 2nQ_4/X_4.$$

where

$$\lambda \equiv y_4/Y_4,$$
$$m \equiv \mathfrak{p}_4/y_4 - \mathfrak{P}_4/Y_4,$$
$$n \equiv \mathfrak{q}_4/y_4 - \mathfrak{Q}_4/Y_4,$$
$$a \equiv \mathfrak{r}_4/y_4 - \mathfrak{R}_4/Y_4 - 2m\mathfrak{P}_4/Y_4,$$
$$b \equiv \mathfrak{s}_4/y_4 - \mathfrak{S}_4/Y_4 - n\mathfrak{P}_4/Y_4 - m\mathfrak{Q}_4/Y_4,$$
$$c \equiv \mathfrak{t}_4/y_4 - \mathfrak{T}_4/Y_4 - 2n\mathfrak{Q}_4/Y_4.$$

Proof.
$$y_i = \lambda Y_i,$$
$$\frac{\partial y_i}{\partial u} = \lambda \frac{\partial Y_i}{\partial u} + Y_i \frac{\partial \lambda}{\partial u}.$$
$$\lambda = y_4/Y_4,$$
$$\frac{\partial \lambda}{\partial u} = \mathfrak{p}_4/Y_4 - \lambda \mathfrak{P}_4/Y_4,$$
$$= \lambda(\mathfrak{p}_4/y_4 - \mathfrak{P}_4/Y_4).$$

Therefore $\quad \mathfrak{p}_i = \lambda(\mathfrak{P}_i + mY_i), \qquad m = \mathfrak{p}_4/y_4 - \mathfrak{P}_4/Y_4.$

Similarly, $\quad \mathfrak{q}_i = \lambda(\mathfrak{Q}_i + nY_i), \qquad n = \mathfrak{q}_4/y_4 - \mathfrak{Q}_4/Y_4.$

$$\bar{y}_i = y_i/Y_i, \qquad i = 1,\ 2,\ 3.$$

$$\frac{\partial \bar{y}_i}{\partial u} = \mathfrak{p}_4/y_4 - y_1\mathfrak{p}_4/y_4^2.$$

$$\frac{\partial^2 \bar{y}_i}{\partial u^2} = \mathfrak{r}_i/y_4 - \mathfrak{p}_i\mathfrak{p}_4/y_4^2 - \mathfrak{p}_i\mathfrak{p}_4/y_4^2 - y_i\mathfrak{r}_4/y_4^2 + 2y_i\mathfrak{p}_4^2/y_4^3,$$

$$\frac{\partial^2 \bar{Y}_i}{\partial u^2} = \mathfrak{R}_i/Y_4 - \mathfrak{P}_i\mathfrak{P}_4/Y_4^2 - \mathfrak{P}_i\mathfrak{P}_4/Y_4^2 - Y_i\mathfrak{R}_4/Y_4^2 + 2Y_i\mathfrak{P}_4^2/Y_4.$$

$$y_i = \bar{y}_i\, y_4,$$

$$\mathfrak{p}_i = \frac{\partial \bar{y}_i}{\partial u} y_4 + \bar{y}_i\,\mathfrak{p}_4,$$

$$\mathfrak{r}_i = \frac{\partial^2 \bar{y}_i}{\partial u^2} y_4 + 2\frac{\partial y_i}{\partial u}\mathfrak{p}_4 + \bar{y}_i\,\mathfrak{r}_4,$$

$$= \frac{\partial^2 \bar{Y}_i}{\partial u^2} y_4 + 2\frac{\partial \bar{y}_i}{\partial u}\mathfrak{p}_4 + \bar{y}_i\,\mathfrak{r}_4,$$

$$= (\mathfrak{R}_i/Y_4 - \mathfrak{P}_i\mathfrak{P}_4/Y_4^2 - \mathfrak{P}_i\mathfrak{P}_4/Y_4^2 - Y_i\,\mathfrak{R}_4/Y_4^2 + 2Y_i\,\mathfrak{P}_4^2/Y_4^3)\,y_4$$
$$+ 2(\mathfrak{p}_i/y_4 - y_i\mathfrak{p}_4/y_4^2)\,\mathfrak{p}_4 + y_i\,\mathfrak{r}_4/y_4,$$
$$= \lambda[\mathfrak{R}_i - 2\mathfrak{P}_i\,\mathfrak{P}_4/Y_4 - Y_i\,\mathfrak{R}_4/Y_4 + 2Y_i\,\mathfrak{P}_4^2/Y_4^2]$$

$$+2\lambda(\mathfrak{p}_i+mY_i)\mathfrak{p}_4/y_4-2\lambda_i\mathfrak{p}_4^2/y_4^2+\lambda Y_i\mathfrak{r}_4/y_4,$$

since $\lambda=y_4/Y_4,\ \mathfrak{p}_i=\lambda(\mathfrak{P}_i+mY_i),\ y_i=\lambda Y_i,$

$$=\lambda[\mathfrak{R}_i+\mathfrak{P}_i(-2\mathfrak{P}_4/Y_4+2\mathfrak{p}_4/y_4)$$
$$+Y_i(-\mathfrak{R}_4/Y_4+2\mathfrak{P}_4^2/y_4^2-2m\mathfrak{p}_4/y_4-2\mathfrak{p}_4^2/y_4^2+\mathfrak{r}_4/y_4)],$$
$$=\lambda[\mathfrak{R}_i+2m\mathfrak{P}_i+\{(\mathfrak{r}_4/y_4-\mathfrak{R}_4/Y_4)+2\mathfrak{P}_4^2/Y_4^2$$
$$-2\mathfrak{p}_4^2/y_4^2+2(\mathfrak{p}_4/y_4-\mathfrak{P}_4/Y_4)\mathfrak{p}_4/y_4\}Y_i],$$
$$=\lambda[\mathfrak{R}_i+2m\mathfrak{P}_i+\{\mathfrak{r}_4/y_4-\mathfrak{R}_4/Y_4-2\mathfrak{P}_4(\mathfrak{p}_4/y_4-\mathfrak{P}_4/Y_4)/Y_4\}Y_i],$$

thus we have

$$\mathfrak{r}_i=\lambda[\mathfrak{R}_i+2m\mathfrak{P}_i+aY_i],$$

where

$$a\equiv\mathfrak{r}_4/y_4-\mathfrak{R}_4/Y_4-2m\mathfrak{P}_4/Y_4.$$

Similarly we may deduce the expressions for $\mathfrak{s}_i$ and $\mathfrak{t}_i$.

§ 3. Correlative Applicability of Two Surfaces.

13. *Definition.* We say a surface generated by the planes

$$y_i=y_i(u,v),\qquad i=1,2,3,4$$

and a surface generated by the points

$$\xi_i=\xi_i(u,v),\qquad i=1,2,3,4$$

to be correlatively applicable in a point u_0, v_0, when there exists a linear integral homogeneous transformation with the coefficients a_{ik}, which transforms the second surface into a surface as the locus of

$$Y_i=\sum_k a_{ik}\xi_k,\qquad i=1,2,3,4,$$

which has with the first surface a contact of the second order in the point u_0, v_0, so that there the equations (VI) are satisfied.

14. Let the quantities π, κ, ρ, etc. correspond for (ξ) to P, Q, R, etc. for (Y) respectively. Then

$$(YPQR)=A(\xi\pi\kappa\rho),\ \text{etc.},$$

where

$$A\equiv|a_{ik}|.$$

$$(y\mathfrak{p}\mathfrak{r}\mathfrak{s})=\lambda^4(Y,\mathfrak{P}+mY,\mathfrak{R}+2m\mathfrak{P}+aY,\mathfrak{S}+m\mathfrak{Q}+n\mathfrak{P}+bY),$$
$$=\lambda^4[(Y\mathfrak{P}\mathfrak{R}\mathfrak{S})+m(Y\mathfrak{P}\mathfrak{R}\mathfrak{Q})],$$

(VII.) $$(y\mathfrak{p}\mathfrak{r}\mathfrak{s})=\lambda^4A[(\xi\pi\rho\sigma)+m(\xi\pi\rho\kappa)].$$

$$(y\mathfrak{r}\mathfrak{q}\mathfrak{f})=\lambda^4(Y, \mathfrak{R}+2m\mathfrak{P}+aY, \mathfrak{Q}+nY, \mathfrak{S}+m\mathfrak{Q}+n\mathfrak{P}+bY),$$

$$=\lambda^4[(Y\mathfrak{R}\mathfrak{Q}\mathfrak{S})+2m(Y\mathfrak{P}\mathfrak{Q}\mathfrak{S})+n(Y\mathfrak{R}\mathfrak{Q}\mathfrak{P})],$$

(VII$_2$) $$(y\mathfrak{r}\mathfrak{q}\mathfrak{f})=\lambda^4 A[(\xi\rho\kappa\sigma)+2m(\xi\pi\kappa\sigma)+n\,\xi\rho\kappa\pi)].$$

$$(\mathfrak{r}\mathfrak{p}\mathfrak{q}\mathfrak{f})=\lambda^4(\mathfrak{R}+2m\mathfrak{P}+aY, \mathfrak{P}+mY, \mathfrak{Q}+nY, \mathfrak{S}+m\mathfrak{Q}+n\mathfrak{P}+bY),$$

$$=\lambda^4(\mathfrak{R}+2m\mathfrak{P}+aY, \mathfrak{P}, \mathfrak{Q}, \mathfrak{S}+m\mathfrak{Q}+n\mathfrak{P}+bY)$$

$$+\lambda^4 m(\mathfrak{R}+2m\mathfrak{P}+aY, \mathfrak{P}, \mathfrak{Q}, \mathfrak{S}+m\mathfrak{Q}+n\mathfrak{P}+bY)$$

$$+\lambda^4 n(\mathfrak{R}+2m\mathfrak{P}+aY, \mathfrak{P}, Y, \mathfrak{S}+m\mathfrak{Q}+n\mathfrak{P}+bY),$$

$$=\lambda^4(\mathfrak{R}\mathfrak{P}\mathfrak{Q}\mathfrak{S})+\lambda^4 a(Y\mathfrak{P}\mathfrak{Q}\mathfrak{S})+\lambda^4 b(\mathfrak{R}\mathfrak{P}\mathfrak{Q}Y)$$

$$+\lambda^4 m(\mathfrak{R}Y\mathfrak{Q}\mathfrak{S})+\lambda^4 m(\mathfrak{P}Y\mathfrak{Q}\mathfrak{S})+\lambda^4 mn(\mathfrak{R}Y\mathfrak{Q}\mathfrak{P})$$

$$+\lambda^4 n(\mathfrak{R}\mathfrak{P}Y\mathfrak{S})+\lambda^4 nm(\mathfrak{R}\mathfrak{P}Y\mathfrak{Q})$$

$$=\lambda^4 A[(\rho\pi\kappa\sigma)+a(\xi\pi\kappa\sigma)+b(\rho\pi\kappa\xi)+m(\rho\xi\kappa\sigma)$$

$$+2m^2(\pi\xi\kappa\sigma)+mn(\rho\xi\kappa\pi)+n(\rho\pi\xi\sigma)+nm(\rho\pi\xi\kappa)].$$

Combining this with (VII$_1$) and (VII$_2$), we have

(VII$_3$) $$(y\mathfrak{p}\mathfrak{q}\mathfrak{f})+m(y\mathfrak{r}\mathfrak{q}\mathfrak{f})+n(y\mathfrak{p}\mathfrak{r}\mathfrak{f})$$

$$=\lambda^4 A\{(\rho\pi\kappa\sigma)+a(\xi\pi\kappa\sigma)+b(\rho\pi\kappa\xi)\}.$$

Replacing u, $\mathfrak{p}$, $\mathfrak{r}$, a, m by v, $\mathfrak{q}$, $\mathfrak{t}$, c and n respectively, we have

(VII$_4$) $$(y\mathfrak{p}\mathfrak{t}\mathfrak{f})=A\lambda^4[(\xi\kappa\tau\sigma)+n\,\xi\kappa\tau\pi)],$$

(VII$_5$) $$(y\mathfrak{t}\mathfrak{p}\mathfrak{f})=A\lambda^4[(\xi\tau\pi\sigma)+2n(\xi\kappa\pi\sigma)+m(\xi\tau\pi\kappa)],$$

(VII$_6$) $$(\mathfrak{t}\mathfrak{q}\mathfrak{p}\mathfrak{f})+n(y\mathfrak{t}\mathfrak{p}\mathfrak{f})+m(y\mathfrak{p}\mathfrak{t}\mathfrak{f})$$

$$=A\lambda^4\{(\tau\kappa\pi\sigma)+c(\xi\kappa\pi\sigma)+b(\tau\kappa\pi\xi)\}.$$

(VIII) $$\begin{cases}(y\mathfrak{p}\mathfrak{q}\mathfrak{r})=A\lambda^4(\xi\pi\kappa\rho),\\(y\mathfrak{p}\mathfrak{q}\mathfrak{f})=A\lambda^4(\xi\pi\kappa\sigma),\\(y\mathfrak{p}\mathfrak{q}\mathfrak{t})=A\lambda^4(\xi\pi\kappa\tau).\end{cases}$$

15. *Theorem 1°.* The necessary and sufficient condition that the two surfaces generated by the points (ξ) and the planes (y) respectively are correlatively applicable in a point, is that in this point (VIII$_i$) ($i=1, 2, 3, \ldots 6$) and (VIII) are satisfied for some system of values of $\lambda\neq 0$, $A\neq 0$, m, n.

Proof.

I. The necessity has been proved in the last article.

II. Sufficiency. The identity

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 & y_i \\ \mathfrak{p}_1 & \mathfrak{p}_2 & \mathfrak{p}_3 & \mathfrak{p}_4 & \mathfrak{p}_i \\ \mathfrak{q}_1 & \mathfrak{q}_2 & \mathfrak{q}_3 & \mathfrak{q}_4 & \mathfrak{q}_i \\ \mathfrak{r}_1 & \mathfrak{r}_2 & \mathfrak{r}_3 & \mathfrak{r}_4 & \mathfrak{r}_i \\ \mathfrak{s}_1 & \mathfrak{s}_2 & \mathfrak{s}_3 & \mathfrak{s}_4 & \mathfrak{s}_i \end{vmatrix} \equiv 0$$

becomes

$$y_i(\mathfrak{pqrs}) - \mathfrak{p}_i(y\mathfrak{qrs}) + \mathfrak{q}_i(y\mathfrak{prs}) - \mathfrak{r}_i(y\mathfrak{pqs}) + \mathfrak{s}_i(y\mathfrak{pqr}) \equiv 0.$$

Therefore

$$\mathfrak{r}_i(y\mathfrak{pqs}) = (\mathfrak{rpqs})y_i + (y\mathfrak{rqs})\mathfrak{p}_i + (y\mathfrak{prs})\mathfrak{q}_i + (y\mathfrak{pqr})\mathfrak{s}_i .$$

$$\mathfrak{r} = \frac{(\mathfrak{rpqs})}{(y\mathfrak{pqs})}y_i + \frac{(y\mathfrak{rqs})}{(y\mathfrak{pqs})}\mathfrak{p}_i + \frac{(y\mathfrak{prs})}{(y\mathfrak{pqs})}\mathfrak{q}_i + \frac{(y\mathfrak{pqr})}{(y\mathfrak{pqs})}\mathfrak{s}_i ,\ (^1)$$

$$= y_i\left[\frac{(\rho\pi\kappa\sigma) + a(\xi\pi\kappa\sigma) + b(\rho\pi\kappa\xi)}{(\xi\pi\kappa\sigma)} - \frac{m(y\mathfrak{rqs}) + n(y\mathfrak{prs})}{(y\mathfrak{pqs})}\right]$$

$$+ \frac{(y\mathfrak{rqs})}{(y\mathfrak{pqs})}\mathfrak{p}_i + \frac{(y\mathfrak{prs})}{(y\mathfrak{pqs})}\mathfrak{q}_i + \frac{(y\mathfrak{pqr})}{(y\mathfrak{qqs})}\mathfrak{s}_i ,$$

$$= y\ \frac{(\rho\pi\kappa\sigma) + a(\xi\pi\kappa\sigma) + b(\rho\pi\kappa\xi)}{(\xi\pi\kappa\sigma)} + (\mathfrak{p}_i - my_i)\frac{(y\mathfrak{rqs})}{(y\mathfrak{pqs})}$$

$$+ (\mathfrak{q}_i - ny_i)\frac{(y\mathfrak{prs})}{(y\mathfrak{pqs})} + \frac{(\xi\pi\kappa\rho)}{(\xi\pi\kappa\sigma)}\mathfrak{s}_i .$$

$$\mathfrak{r}_i = y_i\left[\frac{(\rho\pi\kappa\sigma)}{(\xi\pi\kappa\sigma)} + a\frac{(\xi\pi\kappa\sigma)}{(\xi\pi\kappa\sigma)} + b\frac{(\rho\pi\kappa\xi)}{(\xi\pi\kappa\sigma)}\right]$$

$$+ (\mathfrak{p}_i - my_i)\left[\frac{(\xi\rho\kappa\sigma)}{(\xi\pi\kappa\sigma)} + 2m\frac{(\xi\pi\kappa\sigma)}{(\xi\pi\kappa\sigma)} + \kappa\frac{(\xi\rho\kappa\pi)}{(\xi\pi\kappa\sigma)}\right]$$

$$+ (\mathfrak{q} - ny_i)\left[\frac{(\xi\pi\rho\sigma)}{(\xi\pi\kappa\sigma)} + \rho\frac{(\xi\pi\rho\kappa)}{(\xi\pi\kappa\sigma)}\right] + \frac{(\xi\pi\kappa\rho)}{(\xi\pi\kappa\sigma)}\mathfrak{s}_i$$

that is,

$$[\mathfrak{r}_i - 2m(\mathfrak{p}_i - my_i) - ay_i] = y_i\frac{(\rho\pi\kappa\sigma)}{(\xi\pi\kappa\sigma)} + (\mathfrak{p}_i - my_i)\frac{(\xi\rho\kappa\sigma)}{(\xi\pi\kappa\sigma)}$$

$$+ (\mathfrak{q}_i - ny_i)\frac{(\xi\pi\rho\sigma)}{(\xi\pi\kappa\sigma)} + \{\mathfrak{s}_i - n_1(\mathfrak{q}_i - ny_i) - n(\mathfrak{p}_i - my_i) - by_i\}\frac{(\xi\pi\kappa\rho)}{(\xi\pi\kappa\sigma)}.$$

---

(1) We have assumed that $(y\mathfrak{pqs}) \neq 0$. If $(y\mathfrak{pqs}) \neq 0$, take e.g. $(y\mathfrak{pqr})$ for the denominator.

Compare this with

$$\rho_h \equiv \bar{\xi}_h \frac{(\rho\pi\kappa\sigma)}{(\bar{\xi}\pi\kappa\sigma)} + \pi_h \frac{(\bar{\xi}\rho\kappa\sigma)}{(\bar{\xi}\pi\kappa\sigma)} + \kappa_h \frac{(\bar{\xi}\pi\rho\sigma)}{(\bar{\xi}\pi\kappa\sigma)} + \sigma_h \frac{(\bar{\xi}\pi\kappa\rho)}{(\bar{\xi}\pi\kappa\sigma)}.$$

Determine a_{ih} in such a way, that

$$y_i = \lambda \sum_h a_{ih} \bar{\xi}_h,$$

$$\mathfrak{p}_i - my = \lambda \sum_h a_{ih} \pi_h,$$

$$\mathfrak{q}_i - ny_i = \lambda \sum_h a_{ih} \pi_h,$$

$$\mathfrak{s}_i - m(\mathfrak{q}_i - ny_i) - n(\mathfrak{p}_i - my_i) - by_i = \lambda \sum_h a_{ih} \sigma_h,$$

which is possible, since

$$(\bar{\xi}\pi\kappa\sigma) \neq 0.$$

Then

$$\mathfrak{r}_i - 2m(\mathfrak{p}_i - my_i) - ay_i = \lambda \sum_h a_{ih} \rho_h.$$

Similarly we have

$$\mathfrak{s}_i - 2n(\mathfrak{q}_i - ny_i) - cy_i = \lambda \sum_h a_{ih} \tau_h.$$

From (VIII), we see that

$$| a_{ih} | \equiv A.$$

Let (Y) denote the plane transformed from $(\bar{\xi})$ by the correlation obtained, then we have

$$y_i = \lambda Y_i,$$

$$\mathfrak{p}_i - my_i = \lambda \mathfrak{P}_i, \quad \text{i.e.} \quad \mathfrak{p}_i = \lambda(\mathfrak{P}_i + mY_i),$$

$$\mathfrak{q}_i - ny_i = \lambda \mathfrak{Q}_i, \quad \text{i.e.} \quad \mathfrak{q}_i = \lambda(\mathfrak{Q}_i + nY_i),$$

$$\mathfrak{r}_i - 2m(\mathfrak{p}_i - my_i) - ay_i = \lambda \mathfrak{R}, \quad \text{i.e.} \quad \mathfrak{r}_i = \lambda(\mathfrak{R}_i + 2m\mathfrak{P}_i + aY_i).$$

Similarly

$$\mathfrak{t}_i = \lambda(\mathfrak{T}_i + 2n\mathfrak{Q}_i + cY_i).$$

$$\mathfrak{s}_i - m(\mathfrak{q}_i - ny_i) - n(\mathfrak{p}_i - my_i) - by_i = \lambda \sum_h a_{ih} \sigma_h = \lambda \mathfrak{S}_i,$$

$$\text{i.e.} \quad \mathfrak{s}_i = \lambda(\mathfrak{S}_i + m\mathfrak{Q}_i + n\mathfrak{P}_i + bY_i).$$

Thus we have obtained (VI). Therefore the condition is sufficient.

16. *Theorem* 2°. The necessary and sufficient condition for the correlative applicability of two surfaces (y), $(\bar{\xi})$ is that the expressions

$$\text{(VIII)}\qquad (y\mathfrak{p}\mathfrak{q}\mathfrak{r}),\quad (y\mathfrak{p}\mathfrak{q}\mathfrak{s}),\quad (y\mathfrak{p}\mathfrak{q}\mathfrak{t}),$$

$$\text{(A)}\qquad (y\mathfrak{p}\mathfrak{q}\mathfrak{s})+2\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{r})}(y\mathfrak{p}\mathfrak{q}\mathfrak{s})-\frac{(y\mathfrak{p}\mathfrak{t}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{t})}(y\mathfrak{r}\mathfrak{q}\mathfrak{p}),$$

$$\text{(B)}\qquad (y\mathfrak{t}\mathfrak{p}\mathfrak{s})+2\frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{s})}{(y\mathfrak{q}\mathfrak{p}\mathfrak{t})}(y\mathfrak{q}\mathfrak{p}\mathfrak{s})-\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{s})}{(y\mathfrak{q}\mathfrak{p}\mathfrak{r})}(y\mathfrak{t}\mathfrak{p}\mathfrak{q}),$$

are proportional to

$$(\xi\pi\kappa\rho),\quad (\xi\pi\kappa\sigma),\quad (\xi\pi\kappa\tau),$$

$$(\xi\pi\kappa\sigma)+2\frac{(\xi\pi\kappa\sigma)}{(\xi\pi\kappa\rho)}(\xi\pi\kappa\sigma)-\frac{(\xi\kappa\tau\sigma)}{(\xi\pi\kappa\tau)}(\xi\rho\kappa\pi),$$

$$(\xi\tau\pi\sigma)+2\frac{(\xi\kappa\tau\sigma)}{(\xi\kappa\pi\tau)}(\xi\kappa\pi\sigma)-\frac{(\xi\pi\rho\sigma)}{(\xi\kappa\pi\rho)}(\xi\tau\pi\kappa)$$

respectively.

Proof.

I. Necessity. For $(y\mathfrak{p}\mathfrak{q}\mathfrak{r})$, $(y\mathfrak{p}\mathfrak{q}\mathfrak{s})$, $(y\mathfrak{p}\mathfrak{q}\tau)$, the necessity follows from (VIII) at once.

$$\text{(VII}_1\text{)}\qquad (y\mathfrak{p}\mathfrak{r}\mathfrak{s})=A\lambda^4[(\xi\pi\rho\sigma)+m(\xi\pi\rho\kappa)],$$

$$(y\mathfrak{p}\mathfrak{q}\mathfrak{r})=A\lambda^4(\xi\pi\kappa\rho).$$

Therefore

$$\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{r})}=\frac{(\xi\pi\rho\sigma)}{(\xi\pi\kappa\rho)}+m\frac{(\xi\pi\rho\kappa)}{(\lambda\pi\kappa\rho)}.$$

Therefore

$$\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{r})}-\frac{(\xi\pi\rho\sigma)}{(\xi\pi\kappa\rho)}=-m.$$

$$\text{(VII}_3\text{)}\qquad (y\mathfrak{q}\mathfrak{t}\mathfrak{s})=A\lambda^4[(\xi\kappa\tau\sigma)+n(\xi\kappa\tau\pi)].$$

$$(y\mathfrak{p}\mathfrak{q}\mathfrak{t})=A\lambda^4(\xi\pi\kappa\tau).$$

Therefore

$$\frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{t})}-\frac{(\xi\kappa\tau\sigma)}{(\xi\pi\kappa\tau)}=n.$$

$$\text{(VII}_2\text{)}\qquad (y\mathfrak{r}\mathfrak{q}\mathfrak{s})=\lambda^4A[(\xi\rho\kappa\sigma)+2m(\xi\pi\kappa\sigma)+n(\xi\rho\kappa\pi)],$$

$$=\lambda^4A\Big[(\xi\rho\kappa\sigma)-2\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{s})}{(y\mathfrak{q}\mathfrak{r})}(\xi\pi\kappa\sigma)+2\frac{(\xi\pi\rho\sigma)}{(\xi\pi\kappa\rho)}(\xi\pi\kappa\sigma)$$

$$+\frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{t})}(\xi\kappa\tau\pi)-\frac{(\xi\kappa\tau\sigma)}{(\xi\pi\kappa\tau)}(\xi\kappa\tau\pi)\Big].$$

Therefore

$$
\text{(A)} \qquad (y\mathfrak{r}\mathfrak{q}\mathfrak{f})+2\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{f})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{r})}(y\mathfrak{p}\mathfrak{q}\mathfrak{f})-\frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{f})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{t})}(y\mathfrak{r}\mathfrak{q}\mathfrak{p}),
$$
$$
=\lambda^4 A\left[(\xi\rho\kappa\sigma)+2\frac{(\xi\pi\rho\sigma)}{(\xi\pi\kappa\rho)}(\xi\pi\kappa\sigma)-\frac{(\xi\kappa\tau\sigma)}{(\xi\pi\kappa\tau)}(\xi\rho\kappa\pi)\right].
$$

Similarly from (VII_5),

$$
\text{(B)} \qquad (y\mathfrak{t}\mathfrak{p}\mathfrak{f})+2\frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{f})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{t})}(y\mathfrak{q}\mathfrak{p}\mathfrak{f})-\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{f})}{(y\mathfrak{q}\mathfrak{p}\mathfrak{r})}(y\mathfrak{t}\mathfrak{p}\mathfrak{q}),
$$
$$
=\lambda^4 A\left[(\xi\tau\pi\sigma)+2\frac{(\xi\kappa\tau\sigma)}{(\xi\pi\kappa\tau)}(\xi\kappa\pi\sigma)-\frac{(\xi\pi\rho\sigma)}{(\xi\kappa\pi\rho)}(\xi\tau\pi\kappa)\right].
$$

II. Sufficiency. Conversely, (A), (B) and (VIII) lead to (VII) and (VIII).

17. *Theorem 3°.* In order that two surfaces may be correlatively applicable at a pair of corresponding elements, it is necessary and sufficient that (i) there the asymptotic directions correspond mutually and that (ii)

$$
(\mathrm{C}_1) \qquad \left[\left\{ {11 \atop 1} \right\}'-2\left\{ {12 \atop 2} \right\}'\right]+2\left\{ {11 \atop 2} \right\}'\frac{D'}{D}-\left\{ {22 \atop 1} \right\}'\frac{D}{D''}
$$
$$
=\left[\left\{ {11 \atop 1} \right\}_1-2\left\{ {12 \atop 2} \right\}_1\right]+2\left\{ {11 \atop 2} \right\}_1\frac{D_1'}{D_1}-\left\{ {22 \atop 1} \right\}_1\frac{D_1}{D_1''},
$$
$$
(\mathrm{C}_2) \qquad \left[\left\{ {22 \atop 2} \right\}'-2\left\{ {12 \atop 1} \right\}'\right]+2\left\{ {22 \atop 1} \right\}'\frac{D'}{D''}-\left\{ {11 \atop 2} \right\}'\frac{D''}{D}
$$
$$
=\left[\left\{ {22 \atop 2} \right\}_1-2\left\{ {12 \atop 1} \right\}_1\right]+2\left\{ {22 \atop 1} \right\}_1\frac{D_1'}{D_1''}-\left\{ {11 \atop 2} \right\}_1\frac{D_1''}{D_1},
$$

the suffixes 1 indicating the values for the second surface.

Proof.

$$
\left\{
\begin{aligned}
\frac{\partial^2 y_i}{\partial u^2} &= \left\{ {11 \atop 1} \right\}'\frac{\partial y_i}{\partial u}+\left\{ {11 \atop 2} \right\}'\frac{\partial y_i}{\partial v}-\mathcal{E}y_i,\\
\frac{\partial^2 y_i}{\partial u\partial v} &= \left\{ {12 \atop 1} \right\}'\frac{\partial y_i}{\partial u}+\left\{ {12 \atop 2} \right\}'\frac{\partial y_i}{\partial v}-\mathcal{F}y_i, \qquad i=1,2,3.\\
\frac{\partial^2 y_i}{\partial v^2} &= \left\{ {22 \atop 1} \right\}'\frac{\partial y_i}{\partial u}+\left\{ {22 \atop 2} \right\}'\frac{\partial y_i}{\partial v}-\mathcal{G}y_i, \text{ (1)}
\end{aligned}
\right.
$$

---

(1) See Takasu, l.c.

$$\begin{cases} \dfrac{\partial^2 y_4}{\partial u^2} = \left\{{1\,1 \atop 1}\right\}' \dfrac{\partial y_4}{\partial u} + \left\{{1\,1 \atop 2}\right\}' \dfrac{\partial y_4}{\partial v} - \mathcal{E} y_4 + D, \\ \dfrac{\partial^2 y_4}{\partial u \partial v} = \left\{{1\,2 \atop 1}\right\}' \dfrac{\partial y_4}{\partial u} + \left\{{1\,2 \atop 2}\right\}' \dfrac{\partial y_4}{\partial v} - \mathcal{F} y_4 + D', \\ \dfrac{\partial^2 y_4}{\partial v^2} = \left\{{2\,2 \atop 1}\right\}' \dfrac{\partial y_4}{\partial u} + \left\{{2\,2 \atop 2}\right\}' \dfrac{\partial y_4}{\partial v} - \mathcal{G} y_4 + D''(^1). \end{cases}$$

$$(y\mathfrak{r}\mathfrak{q}\mathfrak{f}) = \left(y \frac{\partial^2 y}{\partial u^2} \frac{\partial y}{\partial v} \frac{\partial^2 y}{\partial u \partial v} \right)$$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ \left\{{1\,1 \atop 1}\right\}' & \left\{{1\,1 \atop 2}\right\}' & -\mathcal{E} & D \\ 0 & 1 & 0 & 0 \\ \left\{{1\,2 \atop 1}\right\}' & \left\{{1\,2 \atop 2}\right\}' & -\mathcal{F} & D' \end{vmatrix} \cdot \begin{vmatrix} \dfrac{\partial y_1}{\partial u} & \dfrac{\partial y_1}{\partial v} & y_1 & 0 \\ \dfrac{\partial y_2}{\partial u} & \dfrac{\partial y_2}{\partial v} & y_2 & 0 \\ \dfrac{\partial y_3}{\partial u} & \dfrac{\partial y_3}{\partial v} & y_3 & 0 \\ \dfrac{\partial y_4}{\partial u} & \dfrac{\partial y_4}{\partial v} & y_4 & 1 \end{vmatrix},$$

$$= \left(\left\{{1\,1 \atop 1}\right\}' D' - \left\{{1\,2 \atop 1}\right\}' D \right) \cdot \bar{\Delta}.$$

Similarly

$$\begin{aligned} (y\mathfrak{r}\mathfrak{q}\mathfrak{f}) &= \left(\left\{{1\,1 \atop 1}\right\}' D' - \left\{{1\,2 \atop 2}\right\}' D \right) \bar{\Delta}, \\ (y\mathfrak{p}\mathfrak{r}\mathfrak{f}) &= \left(\left\{{1\,1 \atop 2}\right\}' D' - \left\{{1\,2 \atop 2}\right\}' D \right) \bar{\Delta}, \\ (y\mathfrak{q}\mathfrak{t}\mathfrak{f}) &= \left(\left\{{1\,2 \atop 1}\right\}' D'' - \left\{{2\,2 \atop 1}\right\}' D' \right) \bar{\Delta}, \\ (y\mathfrak{t}\mathfrak{p}\mathfrak{f}) &= \left(\left\{{2\,2 \atop 2}\right\}' D' - \left\{{2\,1 \atop 2}\right\} D'' \right) \bar{\Delta}, \\ -(y\mathfrak{p}\mathfrak{q}\mathfrak{r}) &= \bar{\Delta} \cdot D, \\ -(y\mathfrak{p}\mathfrak{q}\mathfrak{f}) &= \bar{\Delta} \cdot D', \\ -(y\mathfrak{p}\mathfrak{q}\mathfrak{t}) &= \bar{\Delta} \cdot D'', \end{aligned}$$

$$\begin{aligned} (\xi\rho\kappa\sigma) &= \left(\left\{{1\,1 \atop 1}\right\}_1 D_1' - \left\{{1\,2 \atop 1}\right\}_1 D_1 \right) \Delta_1, \\ (\xi\pi\rho\sigma) &= \left(\left\{{1\,1 \atop 2}\right\}_1 D_1' - \left\{{1\,2 \atop 2}\right\}_1 D_1 \right) \Delta_1, \\ (\xi\kappa\tau\sigma) &= \left(\left\{{1\,2 \atop 1}\right\}_1 D_1'' - \left\{{2\,2 \atop 1}\right\}_1 D_1' \right) \Delta_1, \\ (\xi\tau\pi\sigma) &= \left(\left\{{2\,2 \atop 2}\right\}_1 D_1' - \left\{{2\,1 \atop 2}\right\}_1 D_1'' \right) \Delta_1, \\ -(\xi\pi\kappa\rho) &= \Delta_1 D_1, \\ -(\xi\pi\kappa\sigma) &= \Delta_1 D_1', \\ -(\xi\pi\kappa\tau) &= \Delta_1 D_1''. \end{aligned}$$

Therefore

$$\text{(A)} \qquad (y\mathfrak{r}\mathfrak{q}\mathfrak{f}) + 2\frac{(y\mathfrak{p}\mathfrak{q}\mathfrak{f})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{r})}(y\mathfrak{p}\mathfrak{q}\mathfrak{f}) - \frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{f})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{t})}(y\mathfrak{r}\mathfrak{q}\mathfrak{p})$$

$$= \lambda^4 A \left[(\xi\rho\kappa\sigma) + 2\frac{(\xi\pi\rho\sigma)}{(\xi\pi\kappa\rho)}(\xi\pi\kappa\sigma) - \frac{(\xi\kappa\tau\sigma)}{(\xi\pi\kappa\tau)}(\xi\rho\kappa\pi) \right]$$

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(1) See Takasu, l.c.

becomes

$$\left(\{{}^{11}_{\ 1}\}'D' - \{{}^{12}_{\ 1}\}'D\right)\bar{\Delta} + 2\frac{\left(\{{}^{11}_{\ 2}\}'D' - \{{}^{12}_{\ 2}\}'D\right)\bar{\Delta}}{\bar{\Delta}D}\cdot\bar{\Delta}\cdot D'$$

$$+\frac{\left(\{{}^{12}_{\ 1}\}'D'' - \{{}^{22}_{\ 1}\}'D'\right)\bar{\Delta}}{\bar{\Delta}D''}\cdot\bar{\Delta}\cdot D$$

$$\equiv\left[\{{}^{11}_{\ 1}\}' - 2\{{}^{12}_{\ 2}\}' + 2\{{}^{11}_{\ 2}\}'\frac{D'}{D} - \{{}^{22}_{\ 1}\}'\frac{D}{D''}\right]\bar{\Delta}D'$$

$$=\lambda^4 A\left[\{{}^{11}_{\ 1}\}' - 2\{{}^{12}_{\ 2}\}' + 2\{{}^{11}_{\ 2}\}'\frac{D_1'}{D_1} - \{{}^{22}_{\ 1}\}_1\frac{D_1}{D_1''}\right]\Delta_1\cdot D_1'.$$

Now

$$-(\mathfrak{y}\mathfrak{p}\mathfrak{q}\mathfrak{f})\equiv\bar{\Delta}D' = -\lambda^4 A(\xi\pi\kappa\sigma)\equiv\lambda^4 A\Delta_1 D_1',$$

whence we obtain (C_1).

Similarly, from (B) we obtain (C_2).

Conversely, we may go from (C_1) and (C_2), (VIII) back to (A), (B), (VIII).

Thus (A), (B), (VIII) are together equivalent to (C_1), (C_2), (VIII) taken together.

Hence the result.

§ 4. Condition for Correlative Applicability expressed in terms of the Darboux-Segre Cubic Form and its Dual.

18. *Theorem 4°*. In order that two surfaces may be correlatively applicable at a pair of corresponding elements, it is *necessary* that (i) the asymptotic lines correspond everywhere about those elements and that (ii) at those elements

$$[(\bar{D}_u\,du^3 + 3\bar{D}_v\,du^2\,dv + 3\bar{D}_u''du\,dv^2 + \bar{D}_v''\,dv^3)$$
$$+(hdu+kdv)(Ddu^2+2D'du\,dv+D''dv^2)]$$
$$:[D_{u_1}\,du^3 + 3D_{v_1}\,du^2\,dv + 3D_{u_1}'\,du\,dv^2 + D_{v_1}''\,dv^3)$$
$$+(h_1\,du+k_1\,dv)(D_1\,du^2+2D_1'\,du\,dv+D_1''\,dv^2)]$$
$$=(Ddu^2+2D'\,du\,dv+D''\,dv^2):(D_1\,du^2+2D_1'\,du\,dv+D_1''\,dv^2),$$

where

$$\bar{D}_u\equiv\frac{\partial D}{\partial u} - 2\{{}^{11}_{\ 1}\}'D - 2\{{}^{11}_{\ 2}\}'D',$$

$$\bar{D}_v\equiv\frac{\partial D}{\partial v} - \{{}^{21}_{\ 1}\}'D - \{{}^{11}_{\ 1}\}'D' - \{{}^{21}_{\ 2}\}'D' - \{{}^{11}_{\ 2}\}'D'',$$

$$\bar{D}_u'' \equiv \frac{\partial D''}{\partial u} - 2\left\{{12 \atop 1}\right\}' D' - 2\left\{{12 \atop 2}\right\}' D'',$$

$$\bar{D}_v'' \equiv \frac{\partial D''}{\partial v} - 2\left\{{22 \atop 1}\right\}' D' - 2\left\{{22 \atop 2}\right\}' D'',$$

$$\bar{h} \equiv -\frac{3}{4}\frac{\partial \log \bar{\mathfrak{K}}}{\partial u}, \quad \bar{k} \equiv -\frac{3}{4}\frac{\partial \log \bar{\mathfrak{K}}}{\partial v},$$

$$D_{u_1} \equiv \frac{\partial D_1}{\partial u} - 2\left\{{11 \atop 1}\right\}_1 D_1 - 2\left\{{11 \atop 2}\right\}_1 D_1',$$

$$D_{v_1} \equiv \frac{\partial D_1}{\partial v} - \left\{{21 \atop 1}\right\}_1 D_1 - \left\{{11 \atop 1}\right\}_1 D_1' - \left\{{21 \atop 2}\right\}_1 D_1' - \left\{{11 \atop 2}\right\}_1 D_1',$$

$$D_{u_1}'' \equiv \frac{\partial D_1''}{\partial u} - 2\left\{{12 \atop 1}\right\}_1 D_1' - 2\left\{{12 \atop 2}\right\}_1 D_1'',$$

$$D_{v_1}'' \equiv \frac{\partial D_1''}{\partial v} - 2\left\{{22 \atop 1}\right\}_1 D_1' - 2\left\{{22 \atop 2}\right\}_1 D_1',$$

$$h_1 \equiv -\frac{3}{4}\frac{\partial \log K}{\partial u}, \quad k_1 \equiv -\frac{3}{4}\frac{\partial \log K}{\partial v}.$$

Proof. Suppose the two surfaces to be correlatively applicable at a pair of corresponding elements, then by Art. 17, the asymptotic lines correspond mutually and (C_1) and (C_2) hold.

Since the asymptotic lines correspond mutually everywhere about the elements, we have

$$\frac{D}{D_1} = \frac{D'}{D_1'} = \frac{D''}{D_1''} = \rho, \text{ say.}$$

Also put

$$\left\{{11 \atop 1}\right\}' - \left\{{11 \atop 1}\right\}_1 \equiv \delta\left\{{11 \atop 1}\right\}, \quad \left\{{11 \atop 2}\right\}' - \left\{{11 \atop 2}\right\}_1 \equiv \delta\left\{{11 \atop 2}\right\}, \text{ etc.}$$

The condition that (C_1) and (C_2) have equal values on the two surfaces shows that

$$\frac{\left\{{11 \atop 1}\right\}' D'' - \left\{{22 \atop 1}\right\}' D'}{D''} - 2\frac{\left\{{12 \atop 2}\right\}' D - \left\{{11 \atop 2}\right\}' D'}{D}$$

$$= \frac{\left\{{11 \atop 1}\right\}_1 D_1'' - \left\{{22 \atop 1}\right\}_1 D_1}{D_1''} - 2\frac{\left\{{12 \atop 2}\right\}_1 D_1 - \left\{{11 \atop 2}\right\}_1 D_1'}{D_1}$$

and that

$$\frac{\left\{{22 \atop 2}\right\}' D - \left\{{11 \atop 2}\right\}' D''}{D} - 2\frac{\left\{{12 \atop 1}\right\}' D'' - \left\{{22 \atop 1}\right\} D'}{D''}$$

$$= \frac{\left\{{22 \atop 2}\right\}_1 D_1 - \left\{{11 \atop 2}\right\}_1 D_1''}{D_1} - 2\frac{\left\{{12 \atop 1}\right\}_1 D_1'' - \left\{{22 \atop 1}\right\}_1 D_1'}{D_1''}.$$

Thus, since

$$D : D' : D'' = D_1 : D_1' : D_1'',$$

we have

$$\frac{\delta\left\{ {11 \atop 1} \right\} D'' - \delta\left\{ {22 \atop 1} \right\} D}{D''} = 2\frac{\delta\left\{ {12 \atop 2} \right\} D - \delta\left\{ {11 \atop 2} \right\} D'}{D} \equiv 2\mu, \text{ say},$$

$$\frac{\delta\left\{ {22 \atop 2} \right\} D - \delta\left\{ {11 \atop 2} \right\} D''}{D} = 2\frac{\delta\left\{ {12 \atop 2} \right\} D - \delta\left\{ {22 \atop 1} \right\} D'}{D''} \equiv 2\nu, \text{ say},$$

whence we have

$$\delta\left\{ {11 \atop 1} \right\} - \delta\left\{ {22 \atop 1} \right\} \frac{D}{D''} = 2\mu,$$

$$\left[\delta\left\{ {11 \atop 1} \right\} - 2\mu\right] D'' = \delta\left\{ {22 \atop 1} \right\} D.$$

Similarly

$$\left[\delta\left\{ {12 \atop 2} \right\} - \mu\right] D = \delta\left\{ {11 \atop 2} \right\} D',$$

$$\left[\delta\left\{ {22 \atop 2} \right\} - 2\nu\right] D = \delta\left\{ {11 \atop 2} \right\} D'',$$

$$\left[\delta\left\{ {12 \atop 1} \right\} - \nu\right] D'' = \delta\left\{ {22 \atop 1} \right\} D',$$

so that

$$D : D' : D'' = \left[\delta\left\{ {11 \atop 1} \right\} - 2\mu\right] : \left[\delta\left\{ {12 \atop 1} \right\} - \nu\right] : \delta\left\{ {22 \atop 1} \right\}$$

$$= \delta\left\{ {11 \atop 2} \right\} : \left[\delta\left\{ {12 \atop 2} \right\} - \mu\right] : \left[\delta\left\{ {22 \atop 2} \right\} - 2\nu\right].$$

Putting

$$\frac{\delta\left\{ {11 \atop 1} \right\} - 2\mu}{D} = \frac{\delta\left\{ {12 \atop 1} \right\} - \nu}{D'} = \frac{\delta\left\{ {22 \atop 1} \right\}}{D''} = h,$$

$$\frac{\delta\left\{ {11 \atop 2} \right\}}{D} = \frac{\delta\left\{ {12 \atop 2} \right\} - \mu}{D'} = \frac{\delta\left\{ {22 \atop 2} \right\} - 2\nu}{D''} = k,$$

we have

$$\delta\left\{ {11 \atop 1} \right\} = 2\mu + hD, \quad \delta\left\{ {12 \atop 1} \right\} = \nu + hD', \quad \delta\left\{ {22 \atop 1} \right\} = hD'',$$

$$\delta\left\{ {11 \atop 2} \right\} = kD, \qquad \delta\left\{ {12 \atop 2} \right\} = \mu + kD', \quad \delta\left\{ {22 \atop 2} \right\} = 2\nu + kD''.$$

The first dual Codazzi equation becomes for the first surface

$$\frac{\partial}{\partial v}(\rho D_1) - \frac{\partial}{\partial u}(\rho D_1') + \left\{ {11 \atop 1} \right\}' D' - \left\{ {12 \atop 1} \right\}' D - \left\{ {12 \atop 2} \right\}' D' + \left\{ {11 \atop 2} \right\}' D'' = 0,$$

and the first Codazzi equation becomes for the second surface

$$\frac{\partial}{\partial v} D_1 - \frac{\partial}{\partial u} D_1' + \left\{ {11 \atop 1} \right\}_1 D_1' - \left\{ {12 \atop 1} \right\}_1 D_1 - \left\{ {12 \atop 2} \right\}_1 D_1' + \left\{ {11 \atop 2} \right\}_1 D_1'' = 0.$$

The former becomes

$$\frac{\partial\rho}{\partial v}D_1+\rho\frac{\partial D_1}{\partial v}-\frac{\partial\rho}{\partial u}D_1'-\rho\frac{\partial D_1'}{\partial u}+\cdots\cdots\cdots=0.$$

The latter, multiplied by ρ, and subtracted from the last, gives

$$\frac{\partial\rho}{\partial v}D_1+\rho\frac{\partial D_1}{\partial v}-\frac{\partial\rho}{\partial u}D_1'-\rho\frac{\partial D_1'}{\partial u}+\left\{{11 \atop 1}\right\}'D'-\left\{{12 \atop 1}\right\}'D-\left\{{12 \atop 2}\right\}'D'$$

$$+\left\{{11 \atop 2}\right\}'D''-\rho\frac{\partial D_1}{\partial v}+\rho\frac{\partial D_1'}{\partial u}-\left\{{11 \atop 1}\right\}_1\rho D_1'+\left\{{12 \atop 1}\right\}_1\rho D_1+\left\{{12 \atop 2}\right\}_1\rho D_1'$$

$$-\left\{{11 \atop 2}\right\}_1\rho D_1''$$

$$=D\frac{\partial\log\rho}{\partial v}-D'\frac{\partial\log\rho}{\partial u}+D'\delta\left\{{11 \atop 1}\right\}-D\delta\left\{{12 \atop 1}\right\}-D'\delta\left\{{12 \atop 2}\right\}+D''\delta\left\{{11 \atop 2}\right\}$$

$$=D\frac{\partial\log\rho}{\partial v}-D'\frac{\partial\log\rho}{\partial u}+D'(2\mu+hD)-D(\nu+hD')$$

$$-D'(\mu+kD')+D''kD=0,$$

$$=Dr_v-D'r_u+\mu D'-\nu D+k(DD''-D'^2)=0,$$

where

$$r\equiv\log\rho.$$

Thus

$$D(r_v-\nu+hD'+kD'')-D'(r_u-\mu+hD+kD')=0.$$

Similarly, from the other equation of Codazzi and its dual, we have

$$D'(r_v-\nu+kD''+hD')-D''(r_u-\mu+hD+kD')=0.$$

Since the surface is not developable, that is, since $DD''-D'^2\neq0$, we have

$$(\alpha)\qquad\begin{cases}r_u-\mu+hD+kD'=0,\\ r_v-\nu+kD''+hD'=0.\end{cases}$$

The coefficient of du^3 in our first cubic form is

$$\bar{D}_u+\bar{h}D$$

$$=\frac{\partial D}{\partial u}-2\left\{{11 \atop 1}\right\}'D-2\left\{{11 \atop 2}\right\}'D'-\frac{3}{4}D\left\{\frac{\partial\log(DD''-D'^2)}{\partial u}-\frac{\partial\log(\mathcal{E}\mathcal{G}-\mathcal{F}^2)}{\partial u}\right\}.$$

Similarly, for the second surface, we have

$$\frac{\partial D_1}{\partial u}-2\left\{{11 \atop 1}\right\}_1D_1-2\left\{{11 \atop 2}\right\}_1D_1'-\frac{3}{4}D_1\left\{\frac{\partial\log(D_1D_1''-D_1'^2)}{\partial u}-\frac{\partial\log(E_1G_1-F_1^2)}{\partial u}\right\}.$$

Multiplying the latter by ρ and subtracting from the former, we have

$$\frac{\partial D}{\partial u}-\rho\frac{\partial D_1}{\partial u}-2\delta\left\{{}^{1\,1}_{\;1}\right\}D-2\delta\left\{{}^{1\,1}_{\;2}\right\}D'$$

$$-\frac{3}{4}D\left[\left\{\frac{\partial \log(DD''-D'^2)}{\partial u}-\frac{\partial \log(D_1D_1''-D_1'^2)}{\partial u}\right\}\right.$$

$$\left.-\left\{\frac{\partial \log(\mathcal{E}\mathcal{G}-\mathcal{F}^2)}{\partial u}-\frac{\partial \log(E_1G_1-F_1^2)}{\partial u}\right\}\right].$$

Now

$$\frac{\partial \log\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}{\partial u}=\left\{{}^{1\,1}_{\;1}\right\}'+\left\{{}^{1\,2}_{\;2}\right\}',$$

$$\frac{\partial \log\sqrt{E_1G_1-F_1^2}}{\partial u}=\left\{{}^{1\,1}_{\;1}\right\}_1+\left\{{}^{1\,2}_{\;2}\right\}_1,$$

$$\frac{\partial \log(DD''-D'^2)}{\partial u}=\frac{\partial \log\rho^2(D_1D_1''-D_1'^2)}{\partial u},$$

$$=\frac{2}{\rho}\frac{\partial\rho}{\partial u}+\frac{\partial \log(D_1D_1''-D_1'^2)}{\partial u}.$$

$$\frac{\partial D}{\partial u}=\frac{\partial(\rho D_1)}{\partial u}=D_1\frac{\partial\rho}{\partial u}+\rho\frac{\partial D_1}{\partial u},$$

$$=\frac{D}{\rho}\frac{\partial\rho}{\partial u}+\rho\frac{\partial D_1}{\partial u}.$$

Therefore we have

$$\frac{D}{\rho}\frac{\partial\rho}{\partial u}-2\delta\left\{{}^{1\,1}_{\;1}\right\}D-2\delta\left\{{}^{1\,1}_{\;2}\right\}D'-\frac{3}{4}D\left[2\frac{\rho_u}{\rho}-2\delta\left\{{}^{1\,1}_{\;1}\right\}-2\delta\left\{{}^{1\,2}_{\;2}\right\}\right],$$

i.e.

$$\frac{1}{2}\left[-Dr_u+D(-\delta\left\{{}^{1\,1}_{\;1}\right\}+3\delta\left\{{}^{1\,2}_{\;2}\right\})-4\delta\left\{{}^{1\,1}_{\;2}\right\}D'\right]$$

$$=\frac{1}{2}\left[-Dr_u+D(-2\mu-hD+3\mu+3kD')-4D'kD\right],$$

$$=\frac{D}{2}\left[-r_u+\mu-hD-kD'\right]=0$$

by virtue of (α).

Therefore the ratio of the coefficient of du^3 in the cubic for the first surface to that for the second surface is equal to

$$\rho=D/D_1=D'/D_1'=D''/D_1''.$$

Similarly for the coefficients of dv^3.

The coefficient of $du^2\,dv$ in the first cubic is

$$3\frac{\partial D}{\partial v}-6\left\{ {12 \atop 1} \right\}'D-6\left\{ {12 \atop 2} \right\}'D'-\frac{3}{4}D\left\{\frac{\partial\log(DD''-D'^2)}{\partial v}-\frac{\partial\log(\mathcal{E}\mathcal{G}-\mathcal{F}^2)}{\partial v}\right\}$$
$$-\frac{3}{2}D'\left\{\frac{\partial\log(DD''-D'^2)}{\partial u}-\frac{\partial\log(\mathcal{E}\mathcal{G}-\mathcal{F}^2)}{\partial u}\right\}.$$

Also

$$\frac{\partial\log(\mathcal{E}\mathcal{G}-\mathcal{F}^2)}{\partial u}=2\left\{ {11 \atop 1} \right\}'+2\left\{ {12 \atop 2} \right\}',$$

$$\frac{\partial\log(\mathcal{E}\mathcal{G}-\mathcal{F}^2)}{\partial v}=2\left\{ {22 \atop 2} \right\}'+2\left\{ {12 \atop 1} \right\}'.$$

The coefficient of $du^2\,dv$ in the second cubic is

$$3\frac{\partial D_1}{\partial v}-6\left\{ {12 \atop 1} \right\}D_1-6\left\{ {12 \atop 2} \right\}_1D_1'-\frac{3}{4}D_1\left\{\frac{\partial\log(D_1D_1''-D_1'^2)}{\partial v}-\frac{\partial\log(E_1G_1-F_1^2)}{\partial v}\right\}$$
$$-\frac{3}{2}D_1'\left\{\frac{\partial\log(D_1D_1''-D_1'^2)}{\partial u}-\frac{\partial\log(E_1G_1-F_1^2)}{\partial u}\right\}.$$

Multiplying the second by ρ and subtracting from the first, we have

$$3\frac{D}{\rho}\frac{\partial\rho}{\partial v}-6D\delta\left\{ {12 \atop 1} \right\}-6D'\delta\left\{ {12 \atop 2} \right\}-\frac{3}{4}D\left[\frac{2}{\rho}\frac{\partial\rho}{\partial v}-2\delta\left\{ {22 \atop 2} \right\}-2\delta\left\{ {12 \atop 1} \right\}\right]$$
$$-\frac{3}{2}D'\left[\frac{2}{\rho}\frac{\partial\rho}{\partial u}-2\delta\left\{ {11 \atop 1} \right\}-2\delta\left\{ {12 \atop 2} \right\}\right]$$
$$=\frac{3}{2}\frac{D}{\rho}\frac{\partial\rho}{\partial v}-D\left[-\frac{3}{2}\delta\left\{ {22 \atop 2} \right\}+\frac{9}{2}\delta\left\{ {12 \atop 1} \right\}\right]+D'\left[\frac{3}{\rho}\frac{\partial\rho}{\partial u}-3\delta\left\{ {11 \atop 1} \right\}+3\delta\left\{ {12 \atop 2} \right\}\right]$$
$$=\frac{1}{2}\left[3Dr_v-D\left(-3\delta\left\{ {12 \atop 2} \right\}+9\delta\left\{ {12 \atop 1} \right\}\right)-D'\left(6r_u-6\delta\left\{ {11 \atop 1} \right\}+6\delta\left\{ {12 \atop 2} \right\}\right)\right],$$
$$=\frac{1}{2}\{3Dr_v-D(-6\nu-3kD''+9\nu+9hD')-D'(6r_u-12\mu-6hD+6\mu+6kD')\},$$
$$=\frac{1}{6}\{3D(r_v-\nu+kD''+hD')-6D'(r_u-\mu+hD+kD')\}=0 \text{ by } (\alpha).$$

Similarly for the coefficients of $du\,dv^2$.
Hence the theorem.

19. *Theorem 5°.* When the dual Darboux-Segre cubic form for the first surface and the Darboux-Segre cubic form for the second surface are in the ratio ρ, in which the corresponding second forms of Gauss stand, then the two surfaces are correlatively applicable.

Proof. The condition that the coefficients of du^3 in the cubic forms for the two surfaces are in the ratio ρ, is as above

$$(1) \qquad D[3\delta\{{}^{1\,2}_{\ 2}\}-\delta\{{}^{1\,1}_{\ 1}\}-r_u]=4\delta\{{}^{1\,1}_{\ 2}\}.D'.$$

Similarly with respect to dv^3:—

$$(2) \qquad D''[3\delta\{{}^{1\,2}_{\ 1}\}-\delta\{{}^{2\,2}_{\ 2}\}-r_v]=4\delta\{{}^{2\,2}_{\ 1}\}D'.$$

From the equation of Codazzi and its dual, it follows as above that

$$(3) \quad (r_v-\delta\{{}^{1\,2}_{\ 1}\})D-D'(r_u-\delta\{{}^{1\,1}_{\ 1}\}+\delta\{{}^{1\,2}_{\ 2}\})+\delta\{{}^{1\,1}_{\ 2}\}D''=0,$$

$$(4) \quad (r_u-\delta\{{}^{1\,2}_{\ 2}\})D''-D'(r_v-\delta\{{}^{2\,2}_{\ 2}\}+\delta\{{}^{1\,2}_{\ 1}\})+\delta\{{}^{2\,2}_{\ 1}\}D=0.$$

Also as above

$$\delta\{{}^{1\,1}_{\ 2}\}=kD, \qquad \delta\{{}^{2\,2}_{\ 1}\}=hD'',$$

$$\delta\{{}^{1\,2}_{\ 2}\}=\mu+kD', \qquad \delta\{{}^{1\,2}_{\ 1}\}=\nu+hD'.$$

The equations (1) and (2) become

$$r_u=-kD'-\delta\{{}^{1\,1}_{\ 1}\}+3\mu,$$

$$r_v=-hD'-\delta\{{}^{2\,2}_{\ 2}\}+3\nu,$$

so that (3), (4) become

$$D(2\nu-\delta\{{}^{2\,2}_{\ 2}\}+kD'')-2D'(2\mu-\delta\{{}^{1\,1}_{\ 1}\}+hD)=0,$$

$$2D'(2\nu-\delta\{{}^{2\,2}_{\ 2}\}+kD'')-D''(2\mu-\delta\{{}^{1\,1}_{\ 1}\}+hD)=0,$$

whence, when the coordinates are general and $DD''-D'^2\neq 0$, we have

$$\delta\{{}^{2\,2}_{\ 2}\}=2\nu+kD'',$$

$$\delta\{{}^{1\,1}_{\ 1}\}=2\mu+hD,$$

whence we can return to (C_1) and (C_2).

§ 5. Second Condition for Projective Applicability of Two Surfaces.

20. Replacing the correlation in the preceding process by a collineation, we may prove the following theorems in succession, of which the righthand side is due to Fubini[1].

Theorem 1°. The necessary and sufficient condition for that the two surfaces generated by the planes (y) and (ξ) respectively are pro-

Theorem 1°. The necessary and sufficient condition for that the two surfaces generated by the points (x) and (ξ) respectively are

(1) Loc. cit.

jectively applicable in a plane, is that on this plane

$$(y\mathfrak{p}\mathfrak{r}\mathfrak{s})=A\lambda^4[(\xi\pi\rho\sigma)+m(\xi\pi\rho\kappa)],$$
$$(y\mathfrak{r}\mathfrak{q}\mathfrak{s})=A\lambda^4[(\xi\rho\kappa\sigma)+2m(\xi\pi\kappa\sigma)+n(\xi\rho\kappa\pi)],$$
$$(\mathfrak{r}\mathfrak{p}\mathfrak{q}\mathfrak{s})+m(y\mathfrak{r}\mathfrak{q}\mathfrak{s})+n(y\mathfrak{p}\mathfrak{r}\mathfrak{s})=A\lambda^4[(\rho\pi\kappa\sigma)+a(\xi\pi\kappa\sigma)+b(\rho\pi\kappa\xi)],$$
$$(y\mathfrak{p}\mathfrak{t}\mathfrak{s})=A\lambda^4[(\xi\kappa\tau\sigma)+n(\xi\kappa\tau\pi)],$$
$$(y\mathfrak{t}\mathfrak{p}\mathfrak{s})=A\lambda^4[(\xi\tau\pi\sigma)+2n(\xi\kappa\pi\sigma)+n(\xi\tau\pi\kappa)],$$
$$(\mathfrak{t}\mathfrak{q}\mathfrak{p}\mathfrak{s})+n(y\mathfrak{t}\mathfrak{p}\mathfrak{s})+m(y\mathfrak{q}\mathfrak{t}\mathfrak{s})=A\lambda^4\{(\tau\kappa\pi\sigma)+c(\xi\kappa\pi\sigma)+b(\tau\kappa\pi\xi)\},$$
$$(y\mathfrak{p}\mathfrak{q}\mathfrak{r})=A\lambda^4(\xi\pi\kappa\rho),$$
$$(y\mathfrak{p}\mathfrak{q}\mathfrak{s})=A\lambda^4(\xi\pi\kappa\sigma),$$
$$(y\mathfrak{p}\mathfrak{q}\mathfrak{t})=A\lambda^4(\xi\pi\kappa\tau)$$

are satisfied for some system of values of $\lambda\neq0$, $A\neq0$ m, n[1].

Theorem 2°. The necessary and sufficient condition for the projective applicability of two surfaces is that the experssions

$$(y\mathfrak{p}\mathfrak{q}\mathfrak{r}),\ (y\mathfrak{p}\mathfrak{q}\mathfrak{s}),\ (y\mathfrak{p}\mathfrak{q}\mathfrak{t}),$$
$$(y\mathfrak{r}\mathfrak{q}\mathfrak{s})+2\frac{(y\mathfrak{p}\mathfrak{r}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{r})}(y\mathfrak{p}\mathfrak{q}\mathfrak{s})-\frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{s})}{(y\mathfrak{p}\mathfrak{q}\mathfrak{t})}(y\mathfrak{r}\mathfrak{q}\mathfrak{p}),$$
$$(y\mathfrak{t}\mathfrak{p}\mathfrak{s})+2\frac{(y\mathfrak{q}\mathfrak{t}\mathfrak{s})}{(y\mathfrak{q}\mathfrak{p}\mathfrak{t})}(y\mathfrak{q}\mathfrak{p}\mathfrak{s})-\frac{(y\mathfrak{q}\mathfrak{r}\mathfrak{s})}{(y\mathfrak{q}\mathfrak{p}\mathfrak{r})}(y\mathfrak{t}\mathfrak{p}\mathfrak{q})$$

projectively applicable in a point, is that in this point

$$(xprs)=A\lambda^4[(\xi\pi\rho\sigma)+m(\xi\pi\rho\kappa)],$$
$$(xrqs)=A\lambda^4[(\xi\rho\kappa\sigma)+2m(\xi\pi\kappa\sigma),+n(\xi\rho\kappa\pi)],$$
$$(rpqs)+m(xrqs)+n(xprs)=A\lambda^4[(\rho\pi\kappa\sigma)+a(\xi\pi\kappa\sigma)+b(\rho\pi\kappa\xi)],$$
$$(xpts)=A\lambda^4[(\xi\kappa\tau\sigma)+n(\xi\kappa\tau\pi)],$$
$$(xtps)=A\lambda^4[(\xi\tau\pi\sigma)+2n(\xi\kappa\pi\sigma)+n(\xi\tau\pi\kappa)],$$
$$(tqps)+n(xtps)+m(xqts)=A\lambda^4\{(\tau\kappa\pi\sigma)+c(\xi\kappa\pi\sigma)+b(\tau\kappa\pi\xi)\},$$
$$(xpqr)=A\lambda^4(\xi\pi\kappa\rho),$$
$$(xpqs)=A\lambda^4(\xi\pi\kappa\sigma),$$
$$(xpqt)=A\lambda^4(\xi\pi\kappa\tau)$$

are satisfied for some system of values of $\lambda\neq0$, $A\neq0$, m, n.

Theorem 2°. The necessary and sufficient condition for the projective applicability of two surfaces is that the expressions

$$(xpqr),\ (xpqs),\ (xpqt),$$
$$(xrqs)+2\frac{(xprs)}{(xpqr)}(xpqs)-\frac{(xqts)}{(xpqt)}(xrqp),$$
$$(xtps)+2\frac{(xqts)}{(xqpt)}(xqps)-\frac{(xprs)}{(xqpr)}(xtpq)$$

(1) The notation will be evident.

constructed for one of the two surfaces are proportional to the analogous expressions constructed for the other.

Theorem 3°. In order that two surfaces are projectively applicable in a plane, it is necessary and sufficient that (i) there the asymptotic directions correspond mutually and that (ii) the expressions

$$\left[\left\{ {11 \atop 1} \right\}' - 2\left\{ {12 \atop 2} \right\}'\right] + 2\left\{ {11 \atop 1} \right\}' \frac{D'}{D} - \left\{ {22 \atop 1} \right\}' \frac{D}{D''}$$

and

$$\left[\left\{ {22 \atop 2} \right\}' - 2\left\{ {12 \atop 1} \right\}'\right] + 2\left\{ {22 \atop 1} \right\}' \frac{D'}{D''} - \left\{ {11 \atop 2} \right\}' \frac{D''}{D}$$

have the same values on one of the two surfaces as on the other.

Theorem 4°. In order that two surfaces may be projectively applicable in a plane, it is necessary, that the asymptotic lines correspond everywhere about that plane and that in that plane

$$\begin{aligned}&[(\bar{D}_u du^3 + 3\bar{D}_v du^2 dv + 3\bar{D}_u'' du dv^2 + \bar{D}_v'' dv^3) \\ &\quad + (\bar{h} du + \bar{k} dv)(D du^2 + 2D' du dv + D'' dv^2)] \\ &: [(\bar{D}_{u_1} du^3 + 3\bar{D}_{v_1} du^2 dv + 3\bar{D}_{u_1}'' du dv^2 + \bar{D}_{v_1}'' dv^3) \\ &\quad + (\bar{h}_1 du + \bar{k}_1 dv)(D_1 du^2 + 2D_1' du dv + D_1'' dv^2)]\end{aligned}$$

constructed for one of the two surfaces are proportional to the analogous expressions constructed for the other.

Theorem 3°. In order that two surfaces are projectively applicable in a point, it is necessary and sufficient that (i) there the asymptotic directions correspond mutually and that (ii) the expressions

$$\left[\left\{ {11 \atop 1} \right\} - 2\left\{ {12 \atop 2} \right\}\right] + 2\left\{ {11 \atop 2} \right\} \frac{D'}{D} - \left\{ {22 \atop 1} \right\} \frac{D}{D''}$$

and

$$\left[\left\{ {22 \atop 2} \right\} - 2\left\{ {12 \atop 1} \right\}\right] + 2\left\{ {22 \atop 1} \right\} \frac{D'}{D''} - \left\{ {11 \atop 2} \right\} \frac{D''}{D}$$

have the same values on one of the two surfaces as on the other.

Theorem 4°. In order that two surfaces may be projectively applicable in a point, it is necessary, that the asymptotic lines correspond everywhere about that point and that in that point

$$\begin{aligned}&[(D_u du^3 + 3D_v du^2 dv + 3D_u'' du dv^2 + D_v'' dv^3) \\ &\quad + (h du + k dv)(D du^2 + 2D' du dv + D'' dv^2)] \\ &: [(D_{u_1} du^3 + 3D_{v_1} du^2 dv + 3D_{u_1}'' du dv^2 + D_{v_1}'' dv^3) \\ &\quad + (h_1 du + k_1 dv)(D_1 du^2 + 2D_1' du dv + D_1'' dv^2)]\end{aligned}$$

$$=[Ddu^2+2D'dudv+D''dv^2]$$
$$:[D_1du^2+2D_1'dudv+D_1''dv^2].$$

Theorem 5°. When the dual Darboux-Segre cubic forms on the two surfaces are in the ratio, in which the corresponding second forms of Gauss stand, then the two surfaces are projectively applicable.

$$=[Ddu^2+2D'dudv+D''dv^2]$$
$$:[D_1du^2+2D_1'dudv+D_1''dv^2].$$

Theorem 5°. When the Darboux-Segre cubic forms on the two surfaces are in the ratio, in which the corresponding second forms of Gauss stand, then the two surfaces are projectively applicable.

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A GENERAL VIEW OF THE THEORY OF SUMMABILITY, II,

by

SATORU TAKENAKA, Ôsaka.

I. Preliminary theorems.

1. The following theorem is due to Lebesgue(1).

Lebesgue's theorem (A). Let $f(\xi)$ be an arbitrary function which is limited and integrable(2) in the interval (a, b). Then the necessary and sufficient conditions that the limit of a sequence

$$\int_a^b \Phi_n(\xi) f(\xi) d\xi$$

exists and is equal to zero, as n tends to infinity, are that

i. there exists a positive number M independent of ξ and n such that

$$|\Phi_n(\xi)| < M, \qquad (n=1, 2, \cdots\cdots),$$

for all values of ξ in (a, b) except at a set of points of measure zero;

ii. for every subinterval (α, β) of (a, b), the limit of the sequence

$$\int_\alpha^\beta \Phi_n(\xi) d\xi$$

vanishes as n tends to infinity.

Applying this theorem, we can prove the following theorem:

Theorem I. *If we define a function $f(\xi)$ as in* Lebesgue's *theorem, then the necessary and sufficient conditions that*

$$\lim_{n=\infty} \int_a^b \Phi_n(x, \xi) f(\xi) d\xi$$

exists and is equal to zero for all values of x in the interval (a, b) are that

i. *for a fixed value of x in (a, b), there exists a positive number M independent of x, ξ and n such that*

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(1) Lebesgue, Sur les intégrales singulières, Ann. de Toulouse, ser. 3, vol. I (1909), p. 52. See also Hahn, Über die Darstellung gegebener Funktionen durch singuläre Integrale, I. Mitteilung, Wiener Denkschrift, vol. 93 (1916), p. 3.

(2) Throughout this paper the term "intgrable" means "integrable in the sense of Lebesgue."

$$|\Phi_n(x, \xi)| < M, \qquad (n=1,\ 2,\cdots\cdots),$$

for all values of ξ *in* (a, b), *except at a set of points of measure zero;*

ii. *for every subinterval* (α, β) *of* (a, b) *which does not contain* x,

$$\lim_{n=\infty}\int_a^\beta \Phi_n(x, \xi)\,d\xi = 0, \qquad (a<x<b).$$

To prove the sufficiency of the given conditions, let us divide the given interval into three parts, such as

$$(a, x-h),\ (x-h,\ x+h),\ (x+h, b),$$

where h is a small positive number.

Then, in the first and the last intervals, the function

$$\Phi_n(x, \xi)$$

satisfies the conditions of Lebesgue's theorem, so that we have

$$\text{(1)} \qquad \lim_{n=\infty}\int_a^{x-h} \Phi_n(x, \xi) f(\xi)\,d\xi = 0,$$

and

$$\text{(2)} \qquad \lim_{n=\infty}\int_{x+h}^{b} \Phi_n(x, \xi) f(\xi)\,d\xi = 0.$$

Since $f(\xi)$ is limited and integrable, it is also absolutely integrable. Hence, for a positive number ε however small, there corresponds a positive number δ such that the following inequality holds:

$$\int_{x-h}^{x+h} |f(\xi)|\,d\xi < \frac{\varepsilon}{2M}, \qquad (h<\delta).$$

Now, by the condition (i), we have, for a fixed value of x in (a, b)

$$|\Phi_n(x, \xi)| < M, \qquad (n=1,\ 2,\cdots\cdots)$$

at all points of ξ in (a, b) except at a set of points of measure zero. Tehrefore if we construct a function $\varphi_n(x, \xi)$, such that

$$|\varphi_n(x, \xi)| < M, \qquad (n=1,\ 2,\cdots\cdots)$$

for all values of x and ξ without exceptions, and moreover

$$\varphi_n(x, \xi) = \Phi_n(x, \xi), \qquad (n=1,\ 2,\cdots\cdots)$$

except at a set of points where $\Phi_n(x, \xi)$ does not satisfy the condition (i), we have

$$\left|\int_{x-h}^{x+h} \Phi_n(x, \xi) f(\xi)\,d\xi\right| = \left|\int_{x-h}^{x+h} \varphi_n(x, \xi) f(\xi)\,d\xi\right| < M\int_{x-h}^{x+h} |f(\xi)|\,d\xi,$$

that is,

$$\left| \int_{x-h}^{x+h} \Phi_n(x, \xi) f(\xi) \, d\xi \right| < \frac{\varepsilon}{3}, \ (n=1, \ 2, \cdots\cdots), \ (\delta < h).$$

But from (1) and (2) we can take an integer N such that the inequalities

$$\left| \int_a^{x-h} \Phi_n(x, \xi) f(\xi) \, d\xi \right| < \frac{\varepsilon}{3}$$

and

$$\left| \int_{x+h}^{b} \Phi_n(x, \xi) f(\xi) \, d\xi \right| < \frac{\varepsilon}{3}$$

hold for all integral values of n greater than N.

Therefore we have the following inequality:

$$\left| \int_a^b \Phi_n(x, \xi) f(\xi) \, d\xi \right| \leqslant \left| \int_a^{x-h} \Phi_n(x, \xi) f(\xi) \, d\xi \right| + \left| \int_{x-h}^{x+h} \Phi_n(x, \xi) f(\xi) \, d\xi \right|$$
$$+ \left| \int_{x+h}^{b} \Phi_n(x, \xi) f(\xi) \, d\xi \right| < \varepsilon, \qquad (n > N),$$

which shows the sufficiency of the given conditions.

Next let us show that the conditions are necessary.

By Lebesgue's theorem, it is obvious that the conditions are necessary for the intervals $(a, x-h)$ and $(x+h, b)$, where h is a positive number however small. Therefore we are satisfied to show that the first condition is necessary for the interval $(x-h, \ x+h)$; and to show this, it is sufficient to give an example of $\Phi_n(x, \xi)$ that satisfies the second condition and the first except in the interval $(x-h, \ x+h)$, and for which

$$\lim_{n=\infty} \int_a^b \Phi_n(x, \xi) f(\xi) \, d\xi$$

is not zero for a suitably chosen function $f(\xi)$.

To this end we will take the function

$$\Phi_n(x, \xi) = \frac{1}{2n\pi} \left[\frac{\sin \frac{n}{2} (\xi - x)}{\sin \frac{1}{2} (\xi - x)} \right]^2, \qquad b - a \leqslant 2\pi.$$

Then for every subinterval (α, β) of (a, b) which does not contain x, we have

$$\lim_{n=\infty} \frac{1}{2n\pi} \int_\alpha^\beta \left[\frac{\sin \frac{n}{2} (\xi - x)}{\sin \frac{1}{2} (\xi - x)} \right]^2 d\xi = 0.$$

Let h be a small positive number, then for every value of ξ not contained in $(x-h, x+h)$, we have

$$\frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}(\xi-x)}{\sin\frac{1}{2}(\xi-x)}\right]^2 < \frac{1}{2n\pi\sin^2\frac{1}{2}h} < M.$$

Thus in this example $\Phi_n(x, \xi)$ satisfies the second condition and the first outside the interval $(x-h, x+h)$, but it is easily seen that this does not satisfy the first condition at the neighborhood of x, that is, there exists an interval $(x-h, x+h)$ at every point of which

$$\frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}(\xi-x)}{\sin\frac{1}{2}(\xi-x)}\right]^2 > G, \qquad (n>N),$$

where G is a positive number however great.

And in this example, as was proved by Féjer, we have

$$\lim_{n=\infty}\frac{1}{2n\pi}\int_a^b f(\xi)\left[\frac{\sin\frac{n}{2}(\xi-x)}{\sin\frac{1}{2}(\xi-x)}\right]^2 d\xi = \frac{1}{2}\left[f(x-0)+f(x+0)\right]$$

for any limited and integrable function $f(\xi)$. This shows the necessity of the condition (i).

2. Theorem 2. *If we define a function $f(\xi)$ as in Lebesgue's theorem, then the necessary and sufficient conditions that*

$$\lim_{n=\infty}\int_a^b F_n(x, \xi)f(\xi)\,d\xi$$

exists and is equal to a limited function for all values of x in the interval (a, b) are that

i. *for a fixed value of x in (a, b), there exists a positive number M independent of x, ξ and n such that*

$$|F_n(x, \xi)| < M, \qquad (n=1, 2, \cdots\cdots)$$

at all points of ξ in (a, b) except at a set of points of measure zero;

ii. *for every subinterval (α, β) of (a, b) which does not contain x*

$$\lim_{n=\infty}\int_\alpha^\beta F_n(x, \xi)\,d\xi$$

exists and is equal to a limited function of x.

Choose arbitrary positive integers n_1, n_2 such that

$$\lim_{n=\infty} n_1=\infty, \qquad \lim_{n=\infty}(n_2-n_1)=\infty.$$

Then, by the first condition, we have

$$|F_{n_2}(x, \xi)-F_{n_1}(x, \xi)| < 2M$$

for all values of ξ in (a, b) except at a set of points of measure zero, and from the second condition we have

$$\lim_{n=\infty}\int_{\alpha}^{\beta}\left[F_{n_2}(x, \xi)-F_{n_1}(x, \xi)\right]d\xi=0,$$

where (α, β) is an arbitrary subinterval of (a, b) and does not contain x.

Therefore if we put

$$\Phi_n(x, \xi)=\left[F_{n_2}(x, \xi)-F_{n_1}(x, \xi)\right],$$

this function satisfies the conditions of Theorem 1.

Hence we have

$$\lim_{n=\infty}\int_a^b \Phi_n(x, \xi)f(\xi)\,d\xi=\lim_{n=\infty}\int_a^b\left[F_{n_2}(x, \xi)-F_{n_1}(x, \xi)\right]f(\xi)\,d\xi=0$$

for all values of x in (a, b), or however small ϵ may be given, we can take an integer N for which the following inequality holds:

$$\left|\int_a^b\left[F_{n_2}(x, \xi)-F_{n_1}(x, \xi)\right]f(\xi)\,d\xi\right|<\epsilon, \qquad (n>N),$$

so that we have

$$\left|\left|\int_a^b F_{n_2}(x, \xi)f(\xi)\,d\xi\right|-\left|\int_a^b F_{n_1}(x, \xi)f(\xi)\,d\xi\right|\right|<\epsilon, \qquad (n>N),$$

which shows the sufficiency of the given conditions.

From the above proof the necessity of the conditions is an obvious consequence of Theorem 1.

II. On the limit of a series of functions of the form

$$\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x, \xi)\,u_i(\xi)\,d\xi.$$

3. In this article let us discuss the limit of a series of functions of the form

$$\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x, \xi)\,u_i(\xi)\,d\xi, \qquad (a<x<b).$$

Theorem 3. *The necessary and sufficient conditions that the series of functions*

$$\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x, \xi)\,u_i(\xi)\,d\xi$$

converges to a limited function as n tends to infinity, whenever the sequence $\{u_i(\xi)\}$ of limited and integrable functions converges to a limited function as n tends to infinity(1), *are that*

i. *for a fixed value of x in (a, b), there exists a positive number K independent of x, ξ and n such that*

$$|F_i^{(n)}(x, \xi)| < K, \ (i=1, 2, \cdots\cdots, n), \ (n=1, 2\cdots\cdots)$$

at all points of ξ in (a, b) except at a set of points of measure zero;

ii. *for every subinterval (α, β) of (a, b) which does not contain x*

$$\lim_{n=\infty}\int_\alpha^\beta F_i^{(n)}(x, \xi)\,d\xi \qquad (i=1, 2, \cdots\cdots)$$

exist;

iii. $$\sum_{i=1}^n \int_a^b |F_i^{(n)}(x, \xi)|\,d\xi < M, \qquad (a<x<b), \ (n=1, 2, \cdots\cdots),$$

where M is a positive constant independent of x and n.

Since u's are all limited, there exists a positive number L such that

$$|u_i(\xi)| < L, \qquad (a \leqq \xi \leqq b), \qquad (i=1, 2, \cdots\cdots).$$

Therefore we have

$$\sum_{i=1}^n \left|\int_a^b F_i^{(n)}(x, \xi) u_i(\xi)\,d\xi\right| < L \sum_{i=1}^n \int_a^b |F_i^{(n)}(x, \xi)|\,d\xi < LM,$$

$$(n=1, 2, \cdots\cdots)$$

for all values of x in (a, b).

But by Theorem 2

$$\lim_{n=\infty}\int_a^b F_i^{(n)}(x, \xi) u_i(\xi)\,d\xi, \qquad (a<x<b), \qquad (i=1, 2, \cdots\cdots)$$

exist, hence

$$\lim_{n=\infty}\sum_{i=1}^n \int_a^b F_i^{(n)}(x, \xi) u_i(\xi)\,d\xi$$

must exist for all values of x in the interval (a, b), and moreover this is absolutely convergent.

Thus the sufficiency of the conditions has been proved.

Next let us show the necessity of the given conditions.

(1) As is obvious from the following proof, there is no necessity for the existence of $\lim_{n=\infty} u_n(\xi)$, and the only necessity is that u's are all limited and integrable in the interval (a, b).

As is easily seen from Theorem 2, the conditions (i) and (ii) are necessary.

To show the necessity of the condition (iii), let us assume that this condition is not satisfed for some value x' of x, and let us construct a sequence $\{u_i(\xi)\}$ which satisfies the conditions of the theorem and for which

$$\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x', \xi)\, u_i(\xi)\, d\xi$$

is divergent as n tends to infinity.

Now if we assume

$$\lim_{n=\infty}\sum_{n=1}^{n}\int_a^b |F_i^{(n)}(x', \xi)|\, d\xi = \infty,$$

then for a positive number G greater than 1, there exists an integer N which satisfies the following inequality:

$$\sum_{i=1}^{n}\int_a^b |F_i^{(n)}(x', \xi)|\, d\xi > G^2, \qquad (n>N).$$

Let n_1 be one of such integers and define u's as follows:

$$u_i(\xi) = \frac{1}{G}\text{ sign. } F_i^{(n)}(x', \xi), \ (a<\xi<b), \ (i=1, 2, \cdots\cdots, n_1).$$

Then we have

$$\sum_{i=1}^{n_1}\int_a^b F_i^{(n_1)}(x', \xi)\, u_i(\xi)\, d\xi = \frac{1}{G}\sum_{i=1}^{n_1}\int_a^b F_i^{(n_1)}(x', \xi)\,|\, d\xi > G.$$

But from the condition (i) we have

$$\int_a^b |F_i^{(n)}(x, \xi)|\, d\xi < (b-a)\, K < A(^1), \qquad (a<x<b),$$

where A is a positive constant independent of x and n, whence

$$\sum_{i=1}^{m}\lim_{n=\infty}\int_a^b |F_i^{(n)}(x, \xi)|\, d\xi$$

is limited for all values of x in the interval (a, b). Let this upper limit be A_m. Then we can choose an integar m_1 such that

$$\sum_{i=1}^{n_1}\int_a^b |F_i^{(n)}(x, \xi)|\, d\xi < A_{n_1} + G, \qquad (n>m_1).$$

---

(2) Haar showed that if $\lim_{n=\infty}\int_a^b |K_n(x', \xi)|\, d\xi = \infty$, there exists a continuous function $u(x)$ such that $\lim_{n=\infty}\int_a^b K_n(x', \xi)\, u(\xi)\, d\xi = \infty$. Haar, Zur Theorie der orthogonalen Funktionensysteme, Math. Ann. 69, p. 335. See also Hahn, loc. cit., p. 9.

Next let us choose an integer n_2 greater than m_1 for which the following inequality holds good:

$$\sum_{i=1}^{n_2}\int_a^b |\,F_i^{(n_2)}(x', \xi)\,|\,d\xi > G^4+G^2+G+A_{n_1}(G+1)$$

and define u's as follows:

$$u_i(\xi)=\frac{1}{G^2}\,\text{sign.}\;F_i^{(n_2)}(x', \xi), \qquad (i=n_1+1,\; n_1+2,\cdots\cdots,\; n_2).$$

Then we can prove the following inequalities:

$$\left|\sum_{i=1}^{n_2}\int_a^b F_i^{(n_2)}(x', \xi)u_i(\xi)\,d\xi\right|$$

$$>\left|\sum_{i=n_1+1}^{n_2}\int_a^b F_i^{(n_2)}(x', \xi)u_i(\xi)\,d\xi\right|-\left|\sum_{i=1}^{n_1}\int_a^b F_i^{(n_2)}(x', \xi)u_i(\xi)\,d\xi\right|$$

$$>\frac{1}{G^2}\sum_{i=n_1+1}^{n_2}\int_b^b |\,F_i^{(n_2)}(x', \xi)\,|\,d\xi-\frac{1}{G}\sum_{i=1}^{n_1}\int_a^b |\,F_i^{(n_2)}(x', \xi)\,|\,d\xi$$

$$>\frac{1}{G^2}\left[\sum_{i=1}^{n_2}\int_a^b |\,F_i^{(n_2)}(x', \xi)\,|\,d\xi-\sum_{i=1}^{n_1}\int_a^b |\,F_i^{(n_2)}(x', \xi)\,|\,d\xi\right]$$

$$-\frac{1}{G}\sum_{i=1}^{n_1}\int_a^b |\,F_i^{(n_2)}(x', \xi)\,|\,d\xi$$

$$>\frac{1}{G^2}\Big[G^4+G^2+G+A_{n_1}(G+1)-(A_{n_1}+G)\Big]-\frac{1}{G}\,(A_{n_1}+G),$$

that is,

$$\left|\sum_{i=1}^{n_2}\int_a^b F_i^{(n_2)}(x', \xi)u_i(\xi)\,d\xi\right| > G^2.$$

Continuing in this way, we can construct a sequence of functions such as

$$u_i(\xi)=\frac{1}{G^m}\,\text{sign.}\;F_i^{(n_m)}(x', \xi),\;\; (i=n_{m-1}+1,\; n_{m-1}+2,\cdots\cdots,\; n_m),$$

and

$$\lim_{n=\infty} u_n(\xi)=0,$$

while

$$\left|\sum_{i=1}^{n_m}\int_a^b F_i^{(n_m)}(x', \xi)u_i(\xi)\,d\xi\right| > G^m,$$

that is,

$$\lim_{n=\infty}\left|\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x',\xi)\,u_i(\xi)\,d\xi\right|=\infty.$$

Thus the construction proves that the condition (iii) is necessary.

As a corollary of the above theorem, we can state the following theorem:

Theorem 4. *The necessary and sufficient conditions that the sequence*

$$\sum_{i=1}^{n}\int_a^b F_i(n,\xi)\,u_i(\xi)\,d\xi$$

is convergent as n tends to infinity, where $u_n(\xi)$ is limited and integrable for all values of n, are that

i. *there exists a positive number K independent of ξ and n such that*

$$|F_i(n,\xi)|<K,\ (i=1,\ 2,\cdots\cdots,\ n),\ \ (a<\xi<b)$$

except at a set of points of measure zero;

ii. *for every subinterval (α,β) of (a,b)*

$$\lim_{n=\infty}\int_\alpha^\beta F_i(n,\xi)\,d\xi$$

exists for all values of i;

iii. $$\sum_{i=1}^{n}\int_a^b|F_i(n,\xi)|\,d\xi<M,\qquad (n=1,\ 2,\cdots\cdots)$$

where M is a positive constant independent of n.

III. Further discussion of the series of the last chapter.

4. In this article, let us examine the limit of the series of the last section more closely.

Theorem 5. *Define a sequence $\{F_i^{(n)}(x,\xi)\}$ of limited and integrable (with respect to ξ) functions as follows:*

i. $$F_i^{(n)}(x,\xi),\qquad (i=1,\ 2,\cdots\cdots)$$

converges uniformly to a limited and integrable (with respect to ξ) function

$$F_i(x,\xi),\qquad (i=1,\ 2,\cdots\cdots)$$

for all values of x and ξ in the interval (a,b);

ii. *both* $\lim_{n=\infty}\int_a^b\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi$ *and* $\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x,\xi)\,d\xi$ *exist and are equal to the same limited function;*

iii. $$\sum_{i=1}^{n}\int_a^b | F_i^{(n)}(x,' \xi) | d\xi < M, \quad (n=1, 2, \cdots\cdots), \quad (a<x<b),$$

where M is a positive constant independent of x and n.

Let $\{u_n(\xi)\}$ be a sequence of limited and integrable functions in the interval (a, b), and let

$$\lim_{n=\infty} u_n(\xi), \qquad (a \leqslant \xi \leqslant b)$$

converge uniformly to a limited and integrable function $u(\xi)$, then the following equality holds good:

$$\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x, \xi)\, u_i(\xi)\, d\xi = \lim_{n=\infty}\int_a^b \sum_{i=1}^{n} F_i^{(n)}(x, \xi)\, u(\xi)\, d\xi$$

$$+\sum_{i=1}^{\infty}\int_a^b F_i(x, \xi)\,[u_i(\xi)-u(\xi)]\, d\xi \ (^1).$$

Firstly from the given conditions it can easily be seen that

$$\sum_{i=1}^{n}\int_a^b | F_i(x, \xi) | d\xi$$

is convergent as n tends to infinity for all valucs of x in (a, b).

For, if this is not the case at a point x' of x, then there exists an integer N_1 for which the inequality

$$\sum_{i=1}^{n}\int_a^b | F_i(x', \xi) | d\xi > 2M, \qquad (n \geqslant N_1)$$

holds.

But, since $F_i^{(n)}(x', \xi)$ is absolutely integrable with respect to ξ and satisfies the condition (i), we have

$$\lim_{n=\infty}\int_a^b | F_i^{(n)}(x', \xi) | d\xi = \int_a^b | F_i(x', \xi) | d\xi, \qquad (i=1, 2, \cdots\cdots)(^2),$$

so that if we put

$$\int_a^b | F_i^{(n)}(x', \xi)\, d\xi = \int_a^b | F_i(x', \xi) | d\xi + \varepsilon_i^{(n)}(x'), \qquad (i=1, 2, \cdots\cdots n),$$

we have

$$\lim_{n=\infty} \varepsilon_i^{(n)}(x') = 0, \qquad (i=1, 2, \cdots\cdots).$$

Consequently we can choose an integer N_2 such that

(1) This theorem is valid in the case when $b=x$, that is, $F_i^{(n)}(x, \xi)=0$ for $(\xi>x)$.

(2) See for example Carathéodory, Vorlesungen über reelle Funktionen (1918), p. 444.

$$|\varepsilon_i^{(n)}(x')| < \frac{M}{N_1}, \quad (i=1,\ 2, \cdots\cdots n), \qquad (n \geqq N_2)$$

holds.

Let N be an integer greater than N_1 and N_2, then we have

$$\sum_{i=1}^{n} \int_a^b |F_i^{(n)}(x', \xi)| \, d\xi > \sum_{i=1}^{N_1} \int_a^b |F_i^{(n)}(x', \xi)| \, d\xi$$

$$> \sum_{i=1}^{N_1} \int_a^b |F_i(x', \xi)| \, d\xi - \sum_{i=1}^{N_1} |\epsilon_i^{(n)}(x')|$$

$$> M, \qquad (n \geqq N),$$

which contradicts the assumption (iii); hence

$$\lim_{n=\infty} \sum_{i=1}^{n} \int_a^b |F_i(x, \xi)| \, d\xi$$

must exist for all values of x in the interval (a, b).

Now we will show the theorem. Since

$$\lim_{n=\infty} u_n(\xi) = u(\xi), \qquad (a \leqq \xi \leqq b),$$

if we put

$$u_n(\xi) = u(\xi) + \alpha_n(\xi), \qquad (n=1,\ 2, \cdots\cdots).$$

we have

$$\lim_{n=\infty} \alpha_n(\xi) = 0$$

for all values of ξ in the interval (a, b).

Therefore by Theorem 1 of Part I,

$$\lim_{n=\infty} \sum_{i=1}^{n} \int_a^b F_i(x, \xi) \, \alpha_i(\xi) \, d\xi$$

exists for all values of x in the interval (a, b).

Let this limit be

$$[F(x), \alpha]$$

and put

$$\sum_{i=1}^{n} \int_a^b F_i(x, \xi) \, \alpha_i(\xi) \, d\xi = [F(x), \alpha] + R_n[\alpha(x)], \ (a \leqq x \leqq b).$$

Then

$$\lim_{n=\infty} R_n[\alpha(x)] = 0$$

for all values of x in the interval (a, b).

Similarly if we put

$$F_i^{(n)}(x, \xi) = F_i(x, \xi) + \delta_i^{(n)}(x, \xi), \ (i=1, \ 2, \cdots\cdots n), \ (a < x, \xi < b),$$

we have

$$\lim_{n=\infty} \delta_i^{(n)}(x, \xi) = 0, \qquad (i=1, \ 2, \cdots\cdots)$$

uniformly for all values of x and ξ in the interval (a, b).

Now, calculating step by step, we have the following equalities:

$$\begin{aligned}
\sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, u_i(\xi)\, d\xi &= \sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, [u(\xi) + \alpha_i(\xi)]\, d\xi \\
&= \sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, u(\xi)\, d\xi + \sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, \alpha_i(\xi)\, d\xi \\
&= \sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, u(\xi)\, d\xi + \sum_{i=1}^{m} \int_a^b F_i^{(n)}(x, \xi)\, \alpha_i(\xi)\, d\xi \\
&\quad + \sum_{i=m+1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, \alpha_i(\xi)\, d\xi \\
&= \int_a^b u(\xi) \sum_{i=1}^{n} F_i^{(n)}(x, \xi)\, d\xi + \sum_{i=1}^{m} \int_a^b F_i(x, \xi)\, \alpha_i(\xi)\, d\xi \\
&\quad + \sum_{i=1}^{m} \int_a^b \delta_i^{(n)}(x, \xi)\, \alpha_i(\xi)\, d\xi + \sum_{i=m+1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, \alpha_i(\xi)\, d\xi.
\end{aligned}$$

But since

$$\lim_{n=\infty} \delta_i^{(n)}(x, \xi) = 0, \qquad (i=1, \ 2, \cdots\cdots)$$

uniformly for all values of x and ξ in the interval (a, b), there exists an integer m_1 such that

$$| \delta_i^{(n)}(x, \xi) | < \frac{\epsilon}{2m\,(b-a)K}, \ (i=1, \ 2, \cdots\cdots, \ m), \ (n \geqq m_1),$$

where ε is a positive constant however small and k is the upper limit of $| \alpha_i(x) |$ for all values of x and i, whence we have

$$\left| \sum_{i=1}^{m} \int_a^b \delta_i^{(n)}(x, \xi)\, \alpha_i(\xi)\, d\xi \right| < \frac{\epsilon}{2m\,(b-a)\,K} \sum_{i=1}^{m} \int_a^b | \alpha_i(\xi) |\, d\xi$$

$$< \frac{\varepsilon}{2}, \qquad (a < x < b) \ (n \geqq m_1).$$

Next from the condition (iii) we have

$$\sum_{i=m+1}^{n}\int_a^b |F_i^{(n)}(x,\xi)|\,d\xi < \sum_{i=1}^{n}\int_a^b |F_i^{(n)}(x,\xi)|\,d\xi < M,$$

and by the definition

$$|\alpha_n(\xi)| < \frac{\varepsilon}{2M}, \qquad (n>m_2), \qquad (a<\xi<b),$$

where m_2 is an integer sufficiently large.

Therefore we have

$$\left|\sum_{i=m+1}^{n}\int_a^b F_i^{(n)}(x,\xi)\,\alpha_i(\xi)\,d\xi\right| < \frac{\varepsilon}{2}, \qquad (n>m_2).$$

By the combination of the above results, we can conclude that

$$\left|\sum_{i=1}^{n}\int_a^b F^{(n)}(x,\xi)\,u_i(\xi)\,d\xi - \int_a^b u(\xi)\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi \right.$$
$$\left. -\sum_{i=1}^{m}\int_a^b F_i(x,\xi)\,\alpha_i(\xi)\,d\xi\right| < \varepsilon,$$
$$(a<x<b), \qquad [n>\max(m_1, m_2)],$$

where

$$\lim_{n=\infty} m=\infty.$$

Replacing $\alpha_i(\xi)$ by $u_i(\xi)-u(\xi)$, we have

$$\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x,\xi)\,u_i(\xi)\,d\xi = \lim_{n=\infty}\int_a^b u(\xi)\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi$$
$$+\sum_{i=1}^{\infty}\int_a^b F_i(x,\xi)\,[u_i(\xi)-u(\xi)]\,d\xi,$$

which is the equality that was required to prove.

In the above theorem, if we put

$$F_i(x,\xi)=0, \qquad (i=1,\ 2,\cdots\cdots)$$

uniformly for all values of x and ξ in the interval (a, b), the equality takes the form

$$\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x,\xi)\,u_i(\xi)\,d\xi = \lim_{n=\infty}\int_a^b u(\xi)\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi.$$

5. In the paper mentioned at the foot-note of § 1, Lebesgue proved the following theorem(1):

(1) Lebesgue, loc. cit., p. 69 et seq., see also Hahn, loc. cit., p. 12.

Lebesgue's theorem (B). Let $\varphi_n(x, \xi)$ be limited for all values of x and ξ in the interval (a, b) and be integrable with respect to ξ, and let $f(x)$ be a continuous function of x in (a, b). Then the necessary and sufficient conditions that

$$\lim_{n=\infty}\int_a^b \varphi_n(x, \xi) f(\xi)\, d\xi = f(x)$$

is valid, for all values of x in the interval (a, b), are that

i. for all values of x not contained in the subinterval (α, β) of (a, b), there exists a positive number k independent of x, ξ and n that satisfies

$$|\varphi_n(x, \xi)| < k, \quad (n=1, 2, \cdots\cdots), \quad (\alpha<\xi<\beta)\text{(}^1\text{)}$$

except at a set of points of measure zero;

ii. for every subinterval (α, β) of (a, b) which does not contain x, the equality

$$\lim_{n=\infty}\int_\alpha^\beta \varphi_n(x, \xi)\, d\xi = 0$$

is satisfied;

iii. there exists a positive number M independent of x and n, such that

$$\int_a^b |\varphi_n(x, \xi)|\, d\xi < M, \qquad (n=1, 2, \cdots\cdots), \qquad (a<x<b);$$

iv.
$$\lim_{n=\infty}\int_a^b \varphi_n(x, \xi)\, d\xi = 1$$

for all values of x in the interval (a, b).

By the application of this theorem to that of the last section, we can prove the following theorem:

Theorem 6. *Define a sequence* $\{F_i^{(n)}(x, \xi)\}$ *of limited and integrable (with respect to* ξ*) functions as follows:*

i.
$$F_i^{(n)}(x, \xi), \qquad (i=1, 2, \cdots\cdots)$$

converges uniformly to a limited and integrable (with respect to ξ*) function* $F_i(x, \xi)$ *for all points in the interval* $(a<x, \xi<b)$;

ii. *for all values of* x *not contained in the subinterval* (α, β) *of*

---

(1) This condition is identical with the following:
There exists a positive number independent of x, ξ and n such that

$$|\varphi_n(x, \xi)| < K, \qquad |x-\xi| > \delta > 0,$$

except at a set of points of measure zero, where δ is a positive number however small.

(a, b), there exists a positive number K independent of x, ξ and n such that

$$\left| \sum_{i=1}^{n} F_i^{(n)}(x, \xi) \right| < K, \qquad (\alpha < \xi < \beta), \qquad (n=1,\ 2, \cdots\cdots);$$

iii. *for every subinterval (α, β) of (a, b) which does not contain x*

$$\lim_{n=\infty} \int_a^\beta \sum_{i=1}^{n} F_i^{(n)}(x, \xi)\, d\xi = 0\ ;$$

iv. $$\sum_{i=1}^{n} \int_a^b | F_i^{(n)}(x, \xi) |\, d\xi < M, \qquad (a<x<b), \qquad (n=1,\ 2,\ \cdots\cdots)$$

where M is a positive constant independent of x and n;

v. $$\lim_{n=\infty} \int_a^b \sum_{i=1}^{n} F_i^{(n)}(x, \xi)\, d\xi = \lim_{n=\infty} \sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, d\xi = 1,$$

for all values of x in the interval (a, b).

Let $\{u_n(\xi)\}$ be a sequence of continuous functions having $u(\xi)$ (which is also continuous) *as a limit as n tends to infinity, then*

$$\lim_{n=\infty} \sum_{n=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, u_i(\xi)\, d\xi = u(x) + \sum_{i=1}^{\infty} \int_a^b F_i(x, \xi)\, [u_i(\xi) - u(\xi)]\, d\xi$$

for all values of x in the interval (a, b).

Particularly, if we put

$$F_i(x, \xi) \equiv 0, \qquad (i=1,\ 2, \cdots\cdots), \qquad (a<x, \xi<b),$$

we have the equality

$$\lim_{n=\infty} \sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, u_i(\xi)\, d\xi = u(x), \qquad (a<x<b).$$

6. In this section let us consider a particular case where

$$F_i^{(n)}(x, \xi) = f_i^{(n)}(x-\xi), \qquad (i=1,\ 2, \cdots\cdots n), \qquad (n=1,\ 2, \cdots\cdots),$$
$$(a<x, \xi<b).$$

and

$$\sum_{i=1}^{n} f_i^{(n)}(t), \qquad (a-b<t<b-a), \qquad (n=1,\ 2, \cdots\cdots)$$

is non-negative.

If we put

$$\lim_{n=\infty}\int_{-c_1}^{c_2}\left[\sum_{i=1}^{n} f_i^{(n)}(t)\right]dt=1,$$

where c_1 and c_2 are arbitrary numbers such that

$$a-b<-c_1<0<c_2<b-a,$$

then by a theorem due to Ogura[1], we have

$$\lim_{n=\infty}\int_a^b\sum_{i=1}^{n} f_i(\xi-x)\,u(\xi)\,d\xi=u(x), \qquad (a<x<b)$$

in which $u(x)$ is a continuous function.

Thus Theorem 6 can be modified as follows:

Theorem 7. *Define a sequence $\{f_i^{(n)}(t)\}$ of limited and integrable functions as follows:*

i. $$\sum_{i=1}^{n} f_i^{(n)}(t), \qquad (a-b<t<b-a)$$

is non-negative for all values of n;

ii. $$\lim_{n=\infty} f_i^{(n)}(t)=f_i(t), \qquad (a-b<t<b-a), \qquad (i=1,\ 2,\cdots\cdots),$$

where $f_i^{(n)}$ is a limited and integrable function;

iii. $$\lim_{n=\infty}\int_{-c_1}^{c_2}\sum_{i=1}^{n} f_i^{(n)}(t)=1, \qquad (a-b<-c_1<0<c_2<b-a);$$

$$(i=1,\ 2,\cdots\cdots);$$

iv. $$\sum_{i=1}^{n}\int_a^b |f_i^{(n)}(\xi-x)|\,d\xi<M, \quad (n=1,\ 2,\cdots\cdots), \quad (a<x<b),$$

where M is a positive constant independent of x and n.

Let $\{u_n(\xi)\}$ be a sequence of continuous functions in (a, b) and let

$$\lim_{n=\infty} u_n(\xi)$$

exist and be equal to a continuous function $u(\xi)$. Then we have

$$\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b f_i^{(n)}(\xi-x)\,u_i(\xi)\,d\xi=u(x)+\sum_{i=1}^{\infty}\int_a^b f_i(\xi-x)\,[u_i(\xi)-u(\xi)]\,d\xi.$$

Particularly, if we put

$$f_i(t)=0, \qquad (i=1,\ 2,\cdots\cdots), \qquad (a-b<t<b-a),$$

---

(1) Ogura, On the theory of approximating functions etc., Tôhoku Math. Journal, vol. 16 (1919), p. 112. See also Lebesgue, loc. cit., p. 69 et seg.; Hobson, On a general convergence theorem and etc., Proceedings of London Math. Soc., ser. 2, vol. 6 (1908), p. 349 et seg.

then the above equality becomes

$$\lim_{n=\infty} \sum_{i=1}^{n} \int_a^b \varphi_i^{(n)}(\xi-x)\, u_i(\xi)\, d\xi = u(x).$$

IV. Differentiations.

7. Let $u_n(\xi)$ have continuous differential coefficients of the order m in the interval (a, b), and assume that

$$\lim_{n=\infty} u_n^{(\nu)}(\xi) = u^{(\nu)}(\xi), \qquad (\nu=0,\ 1,\ 2, \cdots\cdots, m),$$

in which $u^{(m)}(\xi)$ is also continuous(1). Then we can put

$$u_i(\xi) = u_i(x) + \sum_{\nu=1}^{m} \frac{(\xi-x)^\nu}{\nu!} \cdot u_i^{(\nu)}(x) + (\xi-x)^m \cdot R\,[u_i(\xi)],\ \ (a \leqq x, \xi \leqq b),$$

where $R\,[u_i(\xi)]$ is a continuous function and

$$\lim_{\xi=x} R\,[u_i(\xi)] = 0.$$

Therefore we have the following equality

$$\begin{aligned}
\sum_{i=1}^{n} \int_a^b F_i^{(n)}(x, \xi)\, u_i(\xi)\, d\xi = {} & \sum_{i=1}^{n} u_i(x) \int_a^b F_i^{(n)}(x, \xi)\, d\xi \\
& + \sum_{i=1}^{n} \sum_{\nu=1}^{m} \frac{u_i^{(\nu)}(x)}{\nu!} \int_a^b (\xi-x)^\nu F_i^{(n)}(x,\xi)\, d\xi \\
& + \sum_{i=1}^{n} \int_a^b (\xi-x)^m \cdot R\,[u_i(\xi)]\, F_i^{(n)}(x, \xi)\, d\xi.
\end{aligned}$$

Now let the sequence $\{F_i^{(n)}(x, \xi)\}$ satisfy the conditions in Theorem 5 and moreover let us replace the condition (ii) of the same theorem by the following:

ii_a. $\displaystyle \lim_{n=\infty} \int_a^b (\xi-x)^\nu \sum_{i=1}^{n} F_i^{(n)}(x, \xi)\, d\xi$

exists, $(a<x<b)$, $(\nu=0,\ 1,\ 2, \cdots\cdots m)$.

Then we can prove the following equalities:

$$\lim_{n=\infty} \sum_{i=1}^{n} u_i(x) \int_a^b F_i^{(n)}(x, \xi)\, d\xi = u(x) \lim_{n=\infty} \int_a^b \sum_{i=1}^{n} F_i^{(n)}(x, \xi)\, d\xi$$

---

(1) $u_n^{(\nu)}(\xi)$ denotes the differential coefficient of $u_n(\xi)$ of the order ν.

$$+\sum_{i=1}^{\infty}[u_i(x)-u(x)]\int_a^b F_i(x,\xi)\,d\xi,$$

$$\lim_{n=\infty}\sum_{i=1}^{n}\sum_{\nu=1}^{m}\frac{u_i^{(\nu)}(x)}{\nu\,!}\int_a^b(\xi-x)^\nu F_i^{(n)}(x,\xi)\,d\xi$$

$$=\lim_{n=\infty}\sum_{\nu=1}^{m}\sum_{i=1}^{n}\frac{u_i^{(\nu)}(x)}{\nu\,!}\int_a^b(\xi-x)^\nu F_i^{(n)}(x,\xi)\,d\xi$$

$$=\sum_{\nu=1}^{m}\frac{u_i^{(\nu)}(x)}{\nu\,!}\lim_{n=\infty}\int_a^b(\xi-x)^\nu\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi$$

$$+\sum_{\nu=1}^{m}\frac{1}{\nu\,!}\sum_{i=1}^{\infty}[u_i^{(\nu)}(x)-u^{(\nu)}(x)]\int_a^b(\xi-x)^\nu F_i(x,\xi)\,d\xi,$$

so that we have the following:

$$\lim_{n=\infty}\left[\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x,\xi)\,u_i(\xi)\,d\xi-\sum_{i=1}^{n}\int_a^b(\xi-x)^m\,.\,R\,[u_i(\xi)]\,F_i^{(n)}(x,\xi)\,d\xi\right]$$

$$=u(x)\lim_{n=\infty}\int_a^b\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi+\sum_{i=1}^{\infty}[u_i(x)-u(x)]\int_a^b F_i(x,\xi)\,d\xi$$

$$+\sum_{\nu=1}^{m}\frac{u^{(\nu)}(x)}{\nu\,!}\lim_{n=\infty}\int_a^b(\xi-x)^\nu\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi$$

$$+\sum_{\nu=1}^{m}\frac{1}{\nu\,!}\sum_{\nu=1}^{\infty}[u_i^{(n)}(x)-u^{(\nu)}(x)]\int_a^b(\xi-x)^\nu F_i(x,\xi)\,d\xi.$$

Next let us assume that $\{F_i^{(n)}(x,\xi)\}$ satisfies the conditions of Theorem 6 (except the fourth) and the following conditions:

$$\lim_{n=\infty}\int_a^b(\xi-x)^\nu\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi=0,\qquad(\nu=0,\ 1,\ 2,\cdots\cdots,\ (m-1)),$$
$$(a<x<b),$$

$$\lim_{n=\infty}\int_a^b(\xi-x)^m\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi=m\,!,\qquad(a<x<b).$$

Then we can prove that

$$\lim_{n=\infty}\left[\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x,\xi)\,u_i(\xi)\,d\xi-\sum_{i=1}^{n}\int_a^b(\xi-x)^m\,R\,[u_i(\xi)]\,F_\nu^{(n)}(x,\xi)\,d\xi\right]$$
$$=u^m(x),\qquad(a<x<b).$$

Since $R[u_i(\xi)]$ is continuous, we have (applying Theorem 6)

$$\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b(\xi-x)^m R[u_i(\xi)]\,F_i^{(n)}(x,\xi)\,d\xi=\lim_{n=\infty} m!\;R[u_n(x)]=0.$$

Thus we have arrived at the following equality:

$$\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b F_i^{(n)}(x,\xi)\,u_i(\xi)\,d\xi=u^{(m)}(x),\qquad (a<x<b).$$

Theorem 8. *Define a sequence $\{F_i^{(n)}(x,\xi)\}$ as follows:*

i. $$\lim_{n=\infty}F_i^{(n)}(x,\xi)=0,\qquad (i=1,\ 2,\cdots\cdots)$$

uniformly for all values of x and ξ in the interval (a,b);

ii. *for all values of x not contained in the subinterval (α,β) of (a,b),*

$$\left|\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\right|<K,$$

where K is a positive constant independent of x, ξ and n;

iii. *for every subinterval (α,β) of (a,b) which does not contain x*

$$\lim_{n=\infty}\int_\alpha^\beta\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi=0\,;$$

iv. $$\sum_{i=1}^{n}\int_a^b|\,F_i^{(n)}(x,\xi)\,|\,d\xi<M,\quad(a<x<b),\quad(n=1,2,\cdots\cdots),$$

where M is a positive constant independent of x and n;

v. $$\lim_{n=\infty}\int_a^b(\xi-x)^\nu\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi=\lim_{n=\infty}\sum_{i=1}^{\infty}\int_a^b(\xi-x)^\nu F_i^{(n)}(x,\xi)\,d\xi=0,$$

$$(\nu=1,\ 2,\cdots\cdots(m-1)),$$

$$\lim_{n=\infty}\int_a^b(\xi-x)^m\sum_{i=1}^{n}F_i^{(n)}(x,\xi)\,d\xi=\lim_{n=\infty}\sum_{i=1}^{n}\int_a^b(\xi-x)^m F_i^{(n)}(x,\xi)\,d\xi=1,$$

for all values of x in the interval (a,b).

Let $u_i(\xi)$ be an arbitrary function which has continuous differential coefficients to the order m for all values of an integer i and satisfies the equalities

$$\lim_{i=\infty}u_i^{(\nu)}(\xi)=u^{(\nu)}(\xi),\qquad(\nu=0,\ 1,\ 2,\cdots\cdots,\ m),$$

in which $u^{(m)}(\xi)$ *is also continuous. Then*

$$\lim_{n=\infty}\sum_{n=1}^{n}\int_a^b F^{(n)}(x,\xi)\,u_i(\xi)\,d\xi = u^{(m)}(x), \qquad (a<x<b).$$

8. Now we will pass to the differentiation of the function considered in § **6.**

Let $f_i^{(n)}(t)$ satisfy the conditions of Theorem 7, and have the continuous derivative $\dfrac{df_i^{(n)}(t)}{dt}$ in $(a-b<t<b-a)$ for all values of i and n $(i\leqq n)$.

Again let $u_n(\xi)$ have the continuous derivative $u'_n(\xi)$ in (a,b) for all values of n and

$$\lim_{n=\infty} u_n(\xi)=u(\xi), \qquad \lim_{n=\infty} u'_n(\xi)=u'(\xi), \qquad (a\leqq\xi\leqq b),$$

where $u'(\xi)$ is also continuous.

Then we have

$$\frac{d}{dx}\left[\int_a^b f_i^{(n)}(\xi-x)\,u_i(\xi)\,d\xi\right]=\int_a^b u_i(\xi)\,\frac{\partial f_i^{(n)}(\xi-x)}{\partial x}\,d\xi,$$

$$(i=1,\ 2,\cdots\cdots n,), \qquad (n=1,\ 2,\cdots\cdots).$$

Since

$$\int_a^b \frac{du_i(x)}{dx} f_i^{(n)}(\xi-x)\,d\xi=[u_i(\xi)f_i^{(n)}(\xi-x)]_a^b-\int_a^b u_i(\xi)\,\frac{\partial}{\partial\xi}f_i^{(n)}(\xi-x)\,d\xi$$

$$=u_i(b)f_i^{(n)}(b-x)-u_i(a)\,f_i^{(n)}(a-x)+\int_a^b u_i(\xi)\,\frac{\partial}{\partial x}f_i^{(n)}(\xi-x)\,d\xi,$$

$$(a<x<b), \qquad (i=1,\ 2,\cdots\cdots n), \qquad (n=1,\ 2,\cdots\cdots),$$

we obtain

$$\frac{d}{dx}\left[\sum_{i=1}^{n}\int_a^b f_i^{(n)}(\xi-x)\,u_i(\xi)\,d\xi\right]=\sum_{i=1}^{n}\int_a^b f_i^{(n)}(\xi-x)\,u'_i(\xi)\,d\xi$$

$$+\sum_{i=1}^{n}[u_i(a)f_i^{(n)}(a-x)-u_i(b)f_i^{(n)}(b-x)], \ (a<x<b).$$

Applying Theorem 7 to the first term of the righthand side of the above equality, we have

$$\lim_{n=\infty}\frac{d}{dx}\left[\sum_{i=1}^{n}\int_a^b f_i^{(n)}(\xi-x)\,u_i(\xi)\,d\xi\right]=u'(x)+\sum_{i=1}^{\infty}\int_a^b f_i(\xi-x)[u_i'(\xi)$$

$$-u'(\xi)]\,d\xi+\lim_{n=\infty}\sum_{i=1}^{n}[u_i(a)f_i^{(n)}(a-x)-u_i(b)f_i^{(n)}(b-x)]$$

uniformly for all values of x in the interval (a, b).

If we assume that

$$\lim_{n=\infty}\sum_{i=1}^{n}f_i^{(n)}(t) \qquad \text{exists for } (a-b<t<b-a),$$

we have the following equalities(¹):

$$\lim_{n=\infty}\sum_{i=1}^{n}u_i(a)f_i^{(n)}(a-x)=u(a)\lim_{n=\infty}\sum_{i=1}^{n}f_i^{(n)}(a-x)$$

$$+\sum_{i=1}^{\infty}f_i(a-x)\,[u_i(a)-u(a)],$$

$$\lim_{n=\infty}\sum_{i=1}^{n}u_i(b)f_i^{(n)}(b-x)=u(b)\lim_{n=\infty}\sum_{i=1}^{n}f_i^{(n)}(b-x)$$

$$+\sum_{i=1}^{\infty}f_i(b-x)\,[u_i(b)-u(b)].$$

Again if we assume that

$$f_i(t)=0, \qquad (a-b<t<b-a), \qquad (i=1,\ 2,\cdots\cdots),$$

we obtain the following equality:

$$\lim_{n=\infty}\frac{d}{dx}\left[\sum_{i=1}^{n}\int_a^b f_i^{(n)}(\xi-x)\,u_i(\xi)\,d\xi\right]=u'(x)+u(a)\lim_{n=\infty}\sum_{i=1}^{n}f_i^{(n)}(a-x)$$

$$-u(b)\lim_{n=\infty}\sum_{i=1}^{n}f_i^{(n)}(b-x), \qquad (a<x<b).$$

Thus we have arrived at the following theorem:

Theorem 9. *Define a sequence $\{f_i^{(n)}(t)\}$ as follows:*

i. $\lim\limits_{n=\infty}f_i^{(n)}(t)=0, \qquad (a-b<t<b-a), \qquad (i=1,\ 2,\cdots\cdots)$

uniformly;

ii. $\sum\limits_{i=1}^{n}f_i^{(n)}(t)$ *is non-negative for all valves of n and t in $(a-b, b-a)$, and*

$$\lim_{n=\infty}\sum_{i=1}^{n}f_i^{(n)}(t), \qquad (a-b<t<b-a)$$

(¹) This can be proved by a theorem due to Kojima, On a generalized Toeplitz's theorem on limit, etc., Tôhoku Math. Journal, vol. 12 (1918), p. 30).

exists ;

iii. $$\lim_{n=\infty}\int_{-c_1}^{c_2}\sum_{i=1}^{n} f_i^{(n)}(t)\,dt = \lim_{n=\infty}\sum_{i=1}^{n}\int_{-c_1}^{c_2} f_i^{(n)}(t)\,dt = 1,$$

$$(a-b< -c_1<0<c_2<b-a)\,;$$

iv. $$\sum_{i=1}^{n}\int_{a}^{b} |f_i^{(n)}(\xi-x)|\,d\xi < M, \quad (a<x<b), \quad (n=1,\ 2,\cdots\cdots)$$

where M is a positive constant independent of x and n.

Let $u_n(\xi)$ be an arbitrary function which has a continuous differential coefficient $u'_n(\xi)$ in (a, b) for all values of an integer n, and let

$$\lim_{n=\infty} u_n(\xi)=u(\xi), \text{ and } \lim_{n=\infty} u'_n(\xi)=u'(\xi), \ (a \leqslant \xi \leqslant b),$$

where $u'(\xi)$ is also continuous. Then we have

$$\lim_{n=\infty}\frac{d}{dx}\left[\sum_{i=1}^{n}\int_{a}^{b} f_i^{(n)}(\xi-x)\,u_i(\xi)\,d\xi\right]=u'(x)+u(a)\lim_{n=\infty}\sum_{i=1}^{n} f_i^{(n)}(a-x)$$

$$-u(b)\lim_{n=\infty}\sum_{i=1}^{n} f_i^{(n)}(b-x), \qquad (a<x<b).$$

March 1922, Minô near Ôsaka.

Über gewisse Infinitesimaloperationen der höheren Operationsstufen,

von

Robert E. Moritz, Seattle, Wash. U. S. A.

Teil III. Die Lehre von den Ratienten[1].

28. *Die unendliche Reihe d r unbestimmten Formen.*

Wir gehen nun zu der Betrachtung der Formen über, die in den höheren Operationsstufen dem bekannten Grenzausdrucke o/o analog sind. Dieses ist der Form nach das Verhältniss erster Stufe des Moduls null-ter Stufe zu sich selbst. Die erweiterte Form wäre demnach in dem Ausdrucke $M_n \overline{\scriptstyle n+1}\, M_n$ zu suchen, d. h. in dem Verhältniss einer beliebigen Ordnung des Moduls des Verhältnisses nächst niederer Ordnung zu sich selbst. In der Tat findet man, wenn man den Ausdruck $M_n \overline{\scriptstyle n+1}\, M_n$ durch bekannte Zeichen ersetzt,

$$[85] \qquad M_n \overline{\scriptstyle n+1}\, M_n = [(O_n)_{-n-1} - (O_n)_{-n-1}]_{n+1} = (O_{-1} - O_{-1})_{n+1} = (O/O)_n,$$

woraus ersichtlich ist, dass der Ausdruck $M_n \overline{\scriptstyle n+1}\, M_n$ das allgemeine Symbol der Unbestimmtheit ist, aus welchem die Form 0/0 durch Spezialisierung hervorgeht. Giebt man n der Reihe nach die Werte -1, 0, 1, so erhält man die bekannten Formen

$$\begin{aligned}
n &= -1, & M_{-1} \overline{\scriptstyle 0}\, M_{-1} &= \infty - \infty, \\
n &= 0, & M_0 \overline{\scriptstyle 1}\, M_0 &= 0/0, \\
n &= 1, & M_1 \overline{\scriptstyle 2}\, M_1 &= 1^\infty.
\end{aligned}$$

29. *Der Differentiationsprozess als Glied einer unendlichen Reihe von Grenzprozessen.* Wir haben nun den Gesichtspunkt erreicht von dem aus sich die in 16. aufgestellte Frage bejahend beantworten lässt. Wie bei der Erweiterung zu verfahren ist, wird klar, sobald wir in der Definitionsgleichung des Differentiationsprozesses

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(1) Obgleich die Grundzüge der Ratientenlehre einschliesslich der Quotientiallehre schon in meiner früheren Arbeit (Generalization of the Differentiation Process, American Journal of Mathematics, Vol. 24, No. 3) entwickelt wurden, sollen sie wegen der dort benutzten schwerverständlichen Darstellungsweise hier aufs Neue durch ein einfacheres Verfahren abgeleitet und zu weiteren Schlussfolgerungen verwertet werden., in der Hoffnung, die schönen Resultate einem grösseren Lesekreise zugänglich zu machen.

$$\frac{dy}{dx} = \lim_{h=0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}, \quad y = f(x)$$

die Zeichen

$$0, \ +, \ -, \ \div,$$

durch die gleichdeutenden Zeichen

$$M_0, \ \underset{0}{+}, \ \underset{0}{-}, \ \underset{1}{-},$$

ersetzen. Die Gleichung erscheint dann in der Form

$$\frac{dy}{dx} = \lim_{h=M_0} \{ [f(x \underset{0}{+} h) \underset{0}{-} f(x)] \underset{1}{-} h \},$$

aus der sich der Weg zur Erweiterung leicht erkennen lässt.

Man setze

$$\frac{d_1y}{d_1x} = \lim_{h=M_1} \{ [f(x \underset{1}{+} h) \underset{1}{-} f(x)] \underset{2}{-} h \},$$

$$\frac{d_2y}{d_2x} = \lim_{h=M_2} \{ [f(x \underset{2}{+} h) \underset{2}{-} f(x)] \underset{3}{-} h \},$$

und ganz allgemein

[86] $$\frac{d_ny}{d_nx} = \lim_{h=M_n} \{ [f(x \underset{n}{+} h) \underset{n}{-} f(x)] \underset{n+1}{-} h \}.$$

Setzt man rechts für h seinen Grenzwert M_n, so geht $f(x \underset{n}{+} h) \underset{n}{-} f(x)$ über in $f(x) \underset{n}{-} f(x) = M_n$, folglich ist der Klammerausdruck von der Form $M_n \underset{n+1}{-} M_n$, und ein jeder der obigen Ausdrücke definirt einen Grenzprozess.

Die Gleichung [86] definirt nun eine unendliche Reihe von Grenzprozessen und zwar ist diese Reihe zweiseitig unendlich, denn es darf ja n ebenso gut negativ wie positiv sein. Der Differentiationsprozess erscheint hiermit als das mit dem Index 0 behaftete Glied einer unendlichen Kette von Grenzprozessen, d. h. es ist

$$\frac{d_0y}{d_0x} = \frac{dy}{dx}.$$

30. *Die Ratienten und ihre gegenseitigen Beziehungen.*

Die Ergebnisse der im vorigen Paragraphen aufgestellten Grenzprozessen wollen wir kurzweg mit den Namen Ratienten belegen, ausführlicher, den Ratienten der Funktion y mit Bezug auf x. $\frac{d_ny}{d_nx}$ ist der

Ratient n-ter Stufe, der Ratient null-ter Stufe fällt mit dem Differentialquotienten überein, die Quotientiallogarithmen werden wir als die Logarithmen der Ratienten erster Stufe wiedererkennen.

Es soll nun gezeigt werden, wie in der Ausführung der Ratientiation einer gegebenen differentiirbarer Funktion zu verfahren ist. Die Bedingung "differentiirbar" ist daher notwendig, weil, wie sich zeigen wird, jede Ratientiation schliesslich auf eine Differentiation zurückgeführt werden kann. Wir zeigen zuerst, dass sich der Ratient $\frac{d_n y}{d_n x}$ einer beliebigen Stufe auf jeden der Ratienten $\frac{d_{n-1} y}{d_{n-1} x}$, $\frac{d_{n+1} y}{d_{n+1} x}$ der beiden benachbarten Stufen zurückführen lässt. Diese Vorwärts- und Rückwärtsführung kann beliebig oft wiederholt werden, folglich ist die Möglichkeit der Zurückführung der Ratientiation von einer Stufe auf jede andere, also auch auf die Differentiation (null-ter Stufe) nachgewiesen.

Die Definitionsgleichung des Ratienten n-ter Stufe ist

$$\frac{d_n y}{d_n x} = \lim_{h=M_n} \{ [f(x \underset{n}{+} h) \underset{n}{-} f(x)] \underset{n+1}{-} h \}.$$

Setzt man hierin nach [75, rechts] und [62]

$$x \underset{n}{+} h = (x_{-1} \underset{n-1}{+} h_{-1})_1$$

und

$$f(x \underset{n}{+} h) \underset{n}{-} f(x) = [f_{-1}(x \underset{n}{+} h) \underset{n-1}{-} f_{-1}(x)]_1$$

wobei eben wie $x_{-1} = \log x$, so auch $f_{-1}(x) = \log f(x)$ bedeutet, so geht

$$f(x \underset{n}{+} h) \underset{n}{-} f(x) \text{ in } \{ f_{-1}(x_{-1} \underset{n-1}{+} h_{-1})_1 \underset{n-1}{-} f_{-1}(x) \}_1$$

über, und nach nochmaliger Anwendung von [75] folgt ebenso

$$f(x \underset{n}{+} h) \underset{n}{-} f(x) \underset{n+1}{-} h = \{ f_{-1}(x_{-1} \underset{n-1}{+} h_{-1})_1 \underset{n-1}{-} f_{-1}(x) \underset{n}{-} h_{-1} \}_1.$$

Vergleicht man nun den letzten Klammerausdruck mit dem Klammerausdruck in der Gleichung, die den Ratienten $(n-1)$-ter Stufe definirt, nämlich

$$\frac{d_{n-1} y}{d_{n-1} x} = \lim_{h=M_{n-1}} \{ f(x \underset{n-1}{+} h) \underset{n-1}{-} f(x) \underset{n}{-} h \},$$

so bemerkt man, dass sich die beiden Ausdrücke nur dadurch unterscheiden, dass im ersten die Grössen mit Indizen behaftet sind. Beobachtet man

noch, dass als sich h der Grenze M_n nähert, h_{-1} sich gleichzeitig der Grenze M_{n-1} nähert, so erhält man schliesslich

[87] $$\frac{d_n y}{d_n x} = \lim_{h_{-1}=M_{n-1}} \{ f_{-1}(x_{-1} \underset{n-1}{+} h_{-1})_1 \underset{n-1}{-} f_{-1}(x) \underset{n}{-} h_{-1} \}_1 = \left(\frac{d_{n-1} y_{-1}}{d_{n-1} x_{-1}} \right)_1 .$$

Setzt man anderseits nach [75 links] und [61]

$$x \underset{n}{+} h = (x_1 \underset{n+1}{+} h_1)_{-1}$$

und

$$f(x \underset{n}{+} h) \underset{n}{-} f(x) = [f_1(x \underset{n}{+} h) \underset{n+1}{-} f_1(x)]_{-1} ,$$

so ist

$$f(x \underset{n}{+} h) \underset{n}{-} f(x) = \{ f_1(x_1 \underset{n+1}{+} h_1)_{-1} \underset{n+1}{-} f_1(x) \}_{-1}$$

und ebenso durch nochmaliger Anwendung von [75]

$$f(x \underset{n}{+} h) \underset{n}{-} f(x) \underset{n+1}{-} h = \{ f_1(x_1 \underset{n+1}{+} h_1)_{-1} \underset{n+1}{-} f_1(x) \underset{n+2}{-} h_1 \}_{-1} .$$

Es ist aber

$$\frac{d_{n+1} y}{d_{n+1} x} = \lim_{h=M_{n+1}} \{ f(x \underset{n+1}{+} h) \underset{n+1}{-} f(x) \underset{n+2}{-} h \},$$

und dieser Klammerausdruck ist gleich dem vorhergehenden, wenn man die darin vorkommenden Grössen mit Indizen behaftet. Da ferner M_n und M_{n+1} den Grössen h und h_1 entsprechenden Grenzen sind, so ist auch

[88] $$\frac{d_n y}{d_n x} = \lim_{h_1=M_{n+1}} \{ f_1(x_1 \underset{n+1}{+} h_1)_{-1} \underset{n+1}{-} f_1(x) \underset{n+2}{-} h_1 \}_{-1} = \left(\frac{d_{n+1} y_1}{d_{n+1} x_1} \right)_{-1} .$$

Die durch Zusammenstellung von [87] und [88] gewonnene Formel

[89] $$\left(\frac{d_{n+1} y_1}{d_{n+1} x_1} \right)_{-1} = \frac{d_n y}{d_n x} = \left(\frac{d_{n-1} y_{-1}}{d_{n-1} x_{-1}} \right)_1$$

lässt sich nun leicht erweitern. [87] auf $\frac{d_{n-1} y}{d_{n-1} x}$ angewandt gibt

$$\frac{d_{n-1} y}{d_{n-1} x} = \left(\frac{d_{n-2} y_{-1}}{d_{n-2} x_{-1}} \right)_1 , \text{ folglich ist } \frac{d_n y}{d_n x} = \left(\frac{d_{n-2} y_{-2}}{d_{n-2} x_{-2}} \right)_2 ,$$

und ebenso gibt [88] auf $\frac{d_{n+1} y}{d_{n+1} x}$ angewandt

$$\frac{d_{n+1}y}{d_{n+1}x}=\left(\frac{d_{n+2}y_1}{d_{n+2}x_1}\right)_{-1}, \text{ oder } \frac{d_n y}{d_n x}=\left(\frac{d_{n+2}y_2}{d_{n+2}x_2}\right)_{-2}.$$

Diese beiden Resultate lassen sich in

[90] $$\left(\frac{d_{n+2}y_2}{d_{n+2}x_2}\right)_{-2}=\frac{d_n y}{d_n x}=\left(\frac{d_{n-2}y_{-2}}{d_{n-2}x_{-2}}\right)_2$$

zusammenstellen.

Durch Fortsetzung des eben benutzten Verfahrens ergibt sich die allgemeine Beziehung zwischen Ratienten beliebiger Stufen

[91] $$\left(\frac{d_{n+k}y_k}{d_{n+k}x_k}\right)_{-k}=\frac{d_n y}{d_n x}=\left(\frac{d_{n-k}y_{-k}}{d_{n-k}x_{-k}}\right)_k,$$

zwei Formeln, die sich in eine vereinigen, wenn man k das doppelte Vorzeichen gibt.

31. *Gegenbeziehungen der Ratienten und Differentialquotienten.*

Setzt man in [91, links] $k=-n$, und in [91, rechts] $n=-n$ und $k=n$, so erhält man

[92] $$\frac{d_n y}{d_n x}=\left(\frac{dy_{-n}}{dx_{-n}}\right)_n, \quad \frac{d_{-n}y}{d_{-n}x}=\left(\frac{dy_n}{dx_n}\right)_{-n}.$$

Setzt man in [91] $n=0$, so erhält man

[93] $$\frac{dy}{dx}=\left(\frac{d_{-n}y_{-n}}{d_{-n}x_{-n}}\right)_n, \quad \frac{dy}{dx}=\left(\frac{d_n y_n}{d_n x_n}\right)_{-n}.$$

Diese beiden Formeln drücken die gegenseitigen Beziehungen zwischen Ratienten und Differentialquotienten aus. [92] zeigt wie sich jeder Ratient durch einen Differentialquotienten, [93] wie sich jeder Differentialquotient durch einen Ratienten beliebiger Stufe berechnen lässt. Hieraus lässt sich schliessen, dass es theoretisch gleichgiltig ist, welche Operationsstufe wir als die Basis der Infinitesimalrechnung ansehen.

32. *Beziehung zwischen Quotientiallogarithmen und Ratienten erster Stufe.*

Spezialisieren wir die Formeln [92] und [93] weiter durch Einsetzung der Werte 1, 2, etc., für n und beachten die Beziehungen

$$\frac{dy_{-1}}{dx_{-1}}=\frac{d\log y}{d\log x}=\frac{x}{y}\cdot\frac{dy}{dx}, \quad \frac{dy_{-2}}{dx_{-2}}=\frac{d\log\,\log y}{d\log\,\log x}=\frac{\log x}{\log y}\cdot\frac{x}{y}\cdot\frac{dy}{dx} \text{ etc.,}$$

$$\frac{dy_1}{dx_1}=\frac{de^y}{de^x}=\frac{e^y}{e^x}\cdot\frac{dy}{dx},\quad \frac{dy_2}{dx_2}=\frac{de^{e^y}}{de^{e^x}}=\frac{e^{e^y}}{e^{e^x}}\cdot\frac{e^y}{e^x}\cdot\frac{dy}{dx} \text{ etc.,}$$

so erhalten wir

[94] $$\frac{d_1y}{d_1x}=e^{\frac{x}{y}\cdot\frac{dy}{dx}},\quad \frac{d_2y}{d_2x}=e^{e^{\frac{\log x}{\log y}\cdot\frac{x}{y}\cdot\frac{dy}{dx}}} \text{ etc.,}$$

[95] $$\frac{d_{-1}y}{d_{-1}x}=\log\left(\frac{e^y}{e^x}\cdot\frac{dy}{dx}\right),\quad \frac{d_{-2}y}{d_{-2}x}=\log\log\left(\frac{e^{e^y}}{e^{e^x}}\cdot\frac{e^y}{e^x}\cdot\frac{dy}{dx}\right) \text{ etc.}$$

$\frac{x}{y}\cdot\frac{dy}{dx}$ ist der Quotientiallogarithmus von y mit Bezug auf x [Teil I, 5]. Die erste dieser Formeln enthält daher die Beziehung zwischen den Quotientiallogarithmen und den Ratienten erster Stufe, nämlich,

[96] $$\frac{d_1y}{d_1x}=\left(\frac{qy}{qx}\right)_1,\quad \text{oder}\quad \frac{qy}{qx}=\left(\frac{d_1y}{d_1x}\right)_{-1}.$$

Diese beiden Formeln sind spezielle Fälle von allgemeineren Formeln, die wir hier nur anführen wollen, ohne uns bei dem Nachweis aufzuhalten. Sie sind

[97] $$\frac{d_ny}{d_nx}=\left(\frac{qy_{-n+1}}{qx_{-n+1}}\right)_n,\quad \text{oder}\quad \frac{qy_n}{qx_n}=\left(\frac{d_{-n+1}y}{d_{-n+1}x}\right)_{n-1}.$$

33. *Ratienten höherer Ordnung.*

Ist $y=f(x)$ eine Funktion von x, so ist gewöhnlich der Ratient von y mit Bezug auf x ebenfalls eine Funktion von x. Den Ratienten dieser Funktion nennen wir den zweiten Ratienten, oder den Ratienten zweiter Ordnung, der ursprünglichen Funktion, und bezeichnen ihn durch $\frac{d_n^2y}{d_nx^2}$; ähnlich kommt man zu dem Begriffe des Ratienten i-ter Ordnung vermöge der formellen Beziehung

[98] $$\frac{d_n^iy}{d_nx^i}=\frac{d_n}{d_nx}\left(\frac{d_n^{i-1}y}{d_nx^{i-1}}\right).$$

Hier sind die Ratientiationen sämmtlich in derselben Stufe. Ist dieses nicht der Fall, so hat man etwa die Bezeichnung

[99] $$\frac{d_{m,n}^2y}{d_{m,n}x^2}=\frac{d_n}{d_nx}\left(\frac{d_my}{d_mx}\right).$$

Den Ratienten zweiter Ordnung kann man auch mittels des zweifachen Grenzprozesses

$$\frac{d_n^2 y}{d_n x^2} = \lim_{h'=M_n} \lim_{h''=M_n} \{ f(x \underset{n}{+} h' \underset{n}{+} h'') \underset{n}{-} f(x \underset{n}{+} h') \underset{n}{-} f(x \underset{n}{+} h'') \underset{n}{+} f(x) \underset{n+1}{-} h' \underset{n+1}{-} h'' \}$$

definiren. Als Grenze dieses Ausdruckes ergibt sich ohne Schwierigkeit

[100] $$\frac{d_n^2 y}{d_n x^2} = \left(\frac{d^2 y_{-n}}{d x_{-n}^2} \right)_n,$$

und hieraus durch vollständige Induktion

[101] $$\frac{d_n^i y}{d_n x^i} = \left(\frac{d^i y_{-n}}{d x_{-n}^i} \right)_n.$$

[100] und [101] sind spezielle Fälle der allgemeineren Formeln

[102] $$\frac{d_n^2 y}{d_n x^2} = \left(\frac{d_{n-k}^2 y_{-k}}{d_{n-k} x_{-k}^2} \right)_k, \quad \frac{d_n^i y}{d_n x^i} = \left(\frac{d_{n-k}^i y_{-k}}{d_{n-k} x_{-k}^i} \right)_k.$$

34. *Distributionsformeln für Ratienten.*

Es seien $u = f(x)$, $v = \varphi(x)$, zwei differentiirbare Funktionen von x, und $y = u \underset{n}{+} v$. Wir wollen den Versuch machen, den Ratienten m-ter Stufe von y auf Ratienten der Teilfunktionen u und v zurückzuführen.

I. Es sei $m > n$, und $m - n = k$.

Nach [92] und [70] ist

$$\frac{d_m (u \underset{n}{+} v)}{d_m x} = \left(\frac{d(u \underset{n}{+} v)_{-m}}{d x_{-m}} \right)_m = \left(\frac{d(u_{-n} + v_{-n})_{-k}}{d x_{-m}} \right)_m.$$

Auch ist

$$d(u_{-n} + v_{-n})_{-k} = \frac{d(u_{-n} + v_{-n})_{-k}}{d(u_{-n} + v_{-n})_{-k+1}} \cdot d(u_{-n} + v_{-n})_{-k+1} = \frac{d(u_{-n} + v_{-n})_{-k+1}}{(u_{-n} + v_{-n})_{-k+1}}.$$

Durch Wiederholung dieser Beziehung wird

$$d(u_{-n} + v_{-n})_{-k} = \frac{d(u_{-n} + v_{-n})}{(u_{-n} + v_{-n})_{-k+1} (u_{-n} + v_{-n})_{-k+2} \cdots (u_{-n} + v_{-n})_0}.$$

Ferner ist

$$d u_{-n} = \frac{d u_{-n}}{d u_{-n-1}} \cdot d u_{-n-1} = u_{-n} \cdot d u_{-n-1}$$

und deshalb auch

$$d u_{-n} = u_{-n} \cdot u_{-n-1} \cdot u_{-n-2} \cdots u_{-m+1} \cdot d u_{-m},$$

und ebenso

$$dv_{-n} = v_{-n} \cdot v_{-n-1} \cdot v_{-n-2} \cdot \cdots \cdot v_{-m+1} \cdot dv_{-m}\,.$$

Setzt man diese Werte in die erste Gleichung ein und beachtet noch, dass nach [92] $\frac{du_{-m}}{dx_{-m}}$ durch $\left(\frac{d_m u}{d_m x}\right)_{-m}$ und $\frac{dv_{-m}}{dx_{-m}}$ durch $\left(\frac{d_m v}{d_m x}\right)_{-m}$ ersetzt werden darf, so erhält man die gesuchte Distributionsformel

[103]

$$\frac{d_m \underset{n}{(u+v)}}{d_m x} = \left[\frac{u_{-n} \cdot u_{-n-1} \cdots u_{-m+1}\left(\frac{d_m u}{d_m x}\right)_{-m} + v_{-n} \cdot v_{-n-1} \cdots v_{-m+1}\left(\frac{d_m v}{d_m x}\right)_{-m}}{(u_{-n}+v_{-n})\,(u_{-n}+v_{-n})_{-1} \cdot \cdots \cdot (u_{-n}+v_{-n})_{-k+1}}\right]_m$$

II. Es sei $m < n$, und $n - m = k$.
Es ist in diesem Falle wie zuvor

$$\frac{d_m \underset{n}{(u+v)}}{d_m x} = \left(\frac{d(u_{-n}+v_{-n})_{-k}}{dx_{-m}}\right)_m .$$

Nun ist aber

$$d(u_{-n}+v_{-n})_k = \frac{d(u_{-n}+v_{-n})_k}{d(u_{-n}+v_{-n})_{k-1}} \cdot d(u_{-n}+v_{-n})_{k-1} = (u_{-n}+v_{-n})_k \cdot d(u_{-n}+v_{-n})_{k-1},$$

woraus durch Wiederholung

$$d(u_{-n}+v_{-n})_k = (u_{-n}+v_{-n})_k\,(u_{-n}+v_{-n})_{k-1} \cdot \cdots \cdot (u_{-n}+v_{-n})_1 \cdot d(u_{-n}+v_{-n}),$$

und es ist ferner

$$du_{-n} = \frac{du_{-m}}{u_{-n+1}u_{-n+2} \cdot \cdots \cdot u_{-m}},$$

$$dv_{-n} = \frac{dv_{-m}}{v_{-n+1}v_{-n+2} \cdot \cdots \cdot v_{-m}}.$$

Diese Werte führen zu der Formel

[104]

$$\frac{d_m \underset{n}{(u+v)}}{d_m x} = \left[(u_{-n}+v_{-n})_1\,(u_{-n}+v_{-n})_2 \cdots (u_{-n}+v_{-n})_k \left(\frac{\left(\frac{d_m u}{d_m x}\right)_{-m}}{u_{-m}u_{-m+1} \cdots u_{-n+1}} + \frac{\left(\frac{d_m v}{d_m x}\right)_{-m}}{v_{-m}v_{-m+1} \cdots v_{-n+1}}\right)\right]_m .$$

III. Es sei $m=n$.

In diesem Falle verlieren die oben abgeleiteten Formeln ihren Sinn, aber nach [92], [70], und [93] ist

$$\frac{d_n(u \underset{n}{+} v)}{d_n x} = \left(\frac{d(u \underset{n}{+} v)_{-n}}{dx_{-n}}\right)_n = \left(\frac{d(u_{-n}+v_{-n})}{dx_{-n}}\right)_n = \left[\left(\frac{d_n u}{d_n x}\right)_{-n} + \left(\frac{d_n v}{d_n x}\right)_{-n}\right]_n$$

und daher

$$[105] \qquad \frac{d_n(u \underset{n}{+} v)}{d_n x} = \frac{d_n u}{d_n x} \underset{n}{+} \frac{d_n v}{d_n x}.$$

Formeln [103] und [104] enthalten die Regeln für die Distribution der Ratientiationen m-ter Stufe von zwei Funktionen, die durch Operationen n-ter Stufe mit einander verknüpft sind. Fällt die Ratientionsstufe mit der Operationsstufe überein, so entsteht der einfachste Fall der Distribution, Formel [105], die der Leibniz'schen Formel für die Differentiation einer Summe von Funktionen analog ist. Selbsverständlich kann man in allen drei Formeln überall $\underset{n}{+}$ durch $\underset{n}{-}$ ersteze n, wenn man gleichzeitig rechts alle Operationszeichen $+$ durch $-$ ersetzt.

Wir bemerken noch, dass sich die Formeln [103], [104], [105] sogleich, auf beliebig viele Funktionen, die durch $\underset{n}{+}$ oder $\underset{n}{-}$ verknüpft sind, ausdehnen lassen. Es gilt allgemein

$$[106] \qquad \frac{d_m(u \underset{n}{+} v \underset{n}{-} w \text{ etc})}{d_m x}$$

$$= \left[\frac{\prod\limits_{i=-n}^{-m+1} u_i \left(\frac{d_m u}{d_m x}\right)_{-m} + \prod\limits_{i=-n}^{-m+1} v_i \left(\frac{d_m v}{d_m x}\right)_{-m} - \prod\limits_{i=-n}^{-m+1} w_i \left(\frac{d_m w}{d_m x}\right)_{-m} + \text{etc}}{\pi\limits_{i=0}^{-k+1} (u_{-n}+v_{-n}-w_{-n}+\text{etc})}\right]_m,$$

$$m=n+k;$$

$$[107] \qquad \frac{d_m(u \underset{n}{+} v \underset{n}{-} w \text{ etc})}{d_m x}$$

$$= \left[\prod_{i=1}^{k} (u_{-n}+v_{-n}-w_{-n}+\text{etc})_i \left(\frac{\left(\frac{d_m u}{d_m x}\right)_{-m}}{\prod\limits_{i=-m}^{-n+1} u_i} + \frac{\left(\frac{d_m v}{d_m x}\right)_{-m}}{\prod\limits_{i=-m}^{-n+1} v_i} - \frac{\left(\frac{d_m w}{d_m x}\right)_{-m}}{\prod\limits_{i=-m}^{-n+1} w_i} + \text{etc}\right)\right]_m,$$

$$n=m+k;$$

[106] $$\frac{d_n(u \underset{n}{+} v \underset{n}{-} w \text{ etc})}{d_n x} = \frac{d_n u}{d_n x} \underset{n}{+} \frac{d_n v}{d_n x} \underset{n}{-} \frac{d_n w}{d_n x} \quad \text{etc.}$$

Die Tragweite der Formeln [103] und [104] tritt am deutlichsten hervor, wenn man die Formeln nach verschiedenen Richtungen hin spezialisiert.

Setzt man $n=0, 1, 2$, so erhält man die Regeln für die Ratientiation von Summen, Produkten und Potenzen von Funktionen.

Setzt man $m=0$, so ergeben sich die Differentiationsregeln für Funktionen die durch Operationen beliebiger positiver oder negativer Operationsstufen verknüpft sind. Da aber [103] der Bedingung $m>n$ unterworfen ist, so muss für $m=0$ n negativ sein. Wir geben darum n in [103] das negative Vorzeichen und erhalten somit

[109] $$\frac{d(u \underset{-n}{+} v)}{dx} = \frac{u_1 u_2 \cdots\cdots u_n \dfrac{du}{dx} + v_1 v_2 \cdots\cdots v_n \dfrac{dv}{dx}}{(u_n + v_n)(u_n + v_n)_1 \cdots\cdots (u_n + v_n)_{n+1}};$$

[110] $$\frac{d(u \underset{n}{+} v)}{dx} = (u_{-n} + v_{-n})\,(u_{-n} + v_{-n})_1 \cdots (u_{-n} + v_{-n})_n$$

$$\cdot\left(\frac{\dfrac{du}{dx}}{u\,u_{-1} \cdots\cdots u_{-n+1}} + \frac{\dfrac{dv}{dx}}{v\,v_{-1} \cdots\cdots v_{-n+1}}\right).$$

Mit $m=1$, kommen wir auf die Quotientiationsregeln zurück. Ist besonders in [103] $m=1$, $n=0$, so erhalten wir

$$\frac{d_1(u+v)}{d_1 x} = \left[\frac{u\left(\dfrac{d_1 u}{d_1 x}\right)_{-1} + v\left(\dfrac{d_1 v}{d_1 x}\right)_{-1}}{u+v}\right]_1.$$

Es ist aber nach [96]

$$\left(\frac{d_1 u}{d_1 x}\right)_{-1} = \frac{qu}{qx}, \quad \left(\frac{d_1 v}{d_1 x}\right)_{-1} = \frac{qv}{qx}, \quad \frac{d_1(u+v)}{d_1 x} = \frac{q(u+v)}{qx},$$

folglich ist

$$\frac{q(u+v)}{qx} = \frac{u\,\dfrac{qu}{qx} + v\,\dfrac{qv}{qx}}{u+v}.$$

Wird in [105] $n=1$ gesetzt, so ergibt sich

$$\frac{d_1(u\,v)}{d_1 x} = \left[\left(\frac{d_1 u}{d_1 x}\right)_{-1} + \left(\frac{d_1 v}{d_1 x}\right)_{-1}\right]_1$$

eine Formel, die sich mit Hilfe von [96] in

$$\frac{q(uv)}{qx} = \frac{qu}{qx} + \frac{qv}{qx}$$

umschreiben lässt.

Ein interessanter Fall entsteht, wenn wir in [104] $n = m + 1$ setzen. Die Formel reduzirt sich dann auf

$$[111] \quad \frac{d_m(d_{\overset{+}{m+1}} v)}{d_m x} = \left(v_{\overset{+}{m+1}} \frac{d_m u}{d_m x}\right) \underset{m}{+} \left(u_{\overset{+}{m+1}} \frac{d_m u}{d_m x}\right) = \left(v \underset{\times}{\overset{m}{}} \frac{d_m u}{d_m x}\right) + \left(u \underset{m}{\overset{\times}{}} \frac{d_m v}{d_m x}\right).$$

Diese Formel ist der Leibniz'schen Formel für die Differentiation eines Produktes analog, aus der durch Einsetzung von $m = 0$ Leibniz'sche Formel

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

hervortritt.

35. *Weitere Ratientiationsformeln.*

Mit Hilfe der Definitionsgleichung [86], der Distributionsformel [111], und den Gegenbeziehungen Operationen verschiedener Operationsstufen [70]—[76], lassen sich mannigfaltige Regeln für die Vereinfachung der Ratientenrechnung ohne grosse Schwierigkeiten ableiten. Wir wollen uns hier nicht mit der Ausführung der Ableitungen weiter aufhalten, sondern werden uns mit der Zusammenstellung der Hauptresultate begnügen. Die Formeln der linken Seite entsprechen den an der rechten Seite angeführten Formeln der Differentiationsrechnung, die aus jenen durch Gleichzetzung von n mit 0 hervorgehen.

[112] $\frac{d_n a}{d_n x} = M_n$ (a : eine Konstante), $\qquad \frac{da}{dx} = 0$;

[113] $\frac{d_n x}{d_n x} = M_{n+1}$, $\qquad \frac{dx}{dx} = 1$;

[114] $\frac{d_n(u \overset{+}{_n} a)}{d_n x} = \frac{d_n u}{d_n x}$, $\qquad \frac{d(u+a)}{dx} = \frac{du}{dx}$;

[115] $\frac{d_n(a_{\overset{+}{n+1}} u)}{d_n x} = a \underset{n+1}{+} \frac{d_n u}{d_n x}$, $\qquad \frac{d(au)}{dx} = a\frac{du}{dx}$;

$$[116]\quad \frac{d_n(u_n \underset{n+2}{\overset{+}{}} k_{n+1})}{d_n x} = \frac{d_n(u^k)_n}{d_n x}$$

$$= k_n \underset{n+1}{+} [u_n \underset{n+2}{+} (k-1)_{n+1}] \underset{n+1}{+} \frac{d_n u}{d_n x}, \qquad \frac{du^k}{dx} = ku^{k-1}\frac{du}{dx};$$

$$[117]\quad \frac{d_n e^u}{d_n x} = e^u \underset{n+1}{+} \frac{d_n u}{d_n x}, \qquad \frac{de^u}{dx} = e^u \frac{du}{dx};$$

$$[118]\quad \frac{d_n \log u}{d_n x} = M_{n+1} \underset{n+1}{-} u \underset{n+1}{+} \frac{d_n u}{d_n x}, \qquad \frac{d \log u}{dx} = \frac{1}{u}\,\frac{du}{dx};$$

$$[119]\quad \frac{d_n x}{d_n y} = M_{n+1} \underset{n+1}{-} \frac{d_n y}{d_n x}, \qquad \frac{dx}{dy} = 1 \div \frac{dy}{dx};$$

$$[120]\quad \frac{d_n u}{d_n x} = \frac{d_n u}{d_n z} \underset{n+1}{+} \frac{d_n z}{d_n x}, \qquad \frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx};$$

$$[121]\quad \frac{d_n f(x, y)}{d_n t} = \left(\frac{\partial_n f}{\partial_n x} \underset{n+1}{+} \frac{d_n x}{d_n t}\right) \underset{n}{+} \left(\frac{\partial_n f}{\partial_n y} \underset{n+1}{+} \frac{d_n y}{d_n t}\right), \qquad \frac{df(x, y)}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

Unter dem Zeichen $\frac{\partial_n f}{\partial_n x}$ (partieller Ratient mit Bezug auf x) ist zu verstehen, dass während der Operation mit Bezug auf x, y zeitweilig als Konstante zu betrachten ist. Auch ist ersichtlich, dass überall $\underset{n}{+}$ durch $\underset{n}{-}$, $\underset{n+1}{+}$ durch $\underset{n+1}{-}$, und $\underset{n+2}{+}$ durch $\underset{n+2}{-}$ ersetzt werden darf ohne dass dadurch die Formeln ihre Gültigkeit verlieren.

36. *Die Anti-ratienten.*

Unter Anti-ratient verstehen wir die Funktion die aus der Umkehrung der Ratientiation hervorgeht. Ist $f(x)$ eine gegebene Funktion von x, so ist allgemein $\frac{d_n f(x)}{d_n x}$ wieder eine Funktion von x, die wir mit $F(x)$ bezeichnen werden. Ist nun $F(x)$ eine gegebene Funktion, so entsteht die Frage $f(x)$ zu bestimmen so dass $\frac{d_n f(x)}{d_n x} = F(x)$ ist. Die Operation, die von $F(x)$ auf $f(x)$ führt nennen, wir die Anti-ratientiation, und kennzeichnen sie durch das Symbol ${}_n\!\int F(x)\, d_n x$. Es ist somit

[122] $$\int_n F(x)\, d_n x = f(x),$$

worin $F(x)$ so zu bestimmen ist, dass $\dfrac{d_n f(x)}{d_n x} = F(x)$ ist.

Durch Vermittelung der Definitionsgleichung

$$\int_n F(x) \,.\, d_n x = \int_n \frac{d_n f(x)}{d_n x}\, d_n x = f(x)$$

lässt sich aus jeder Ratientiationsformel eine entsprechende Anti-ratientiationsformel ableiten. Den Formeln [112] bis [118] entsprechend gelten die Formeln

[123] $$\int_n M_n d_n x = a \text{ (eine Konstante)},$$

[124] $$\int_n M_{n+1}\, dx_n = x \overset{+}{_n} a,$$

[125] $$\int_n \frac{d_n u}{d_n x} \,.\, d_n x = u \overset{+}{_n} a,$$

[126] $$\int_n \left(a \overset{+}{_{n+1}} \frac{d_n u}{d_n x}\right) d_n x = a \overset{+}{_{n+1}} \int_n \frac{d_n u}{d_n x} \,.\, d_n x,$$

[127] $$\int_n \left[(u_n \overset{+}{_{n+2}} k)_{n+1}^{+} \frac{d_n u}{d_n x}\right] d_n x = u_n \overset{+}{_{n+2}} (k+1) \overset{-}{_{n+1}} (k+1) \overset{+}{_n} a,$$

[128] $$\int_n \left(e^u \overset{+}{_{n+1}} \frac{d_n u}{d_n x}\right) d_n x = e^u \overset{+}{_n} a,$$

[129] $$\int_n \left(M_{n+1} \overset{-}{_{n+1}} u \overset{+}{_{n+1}} \frac{d_n u}{d_n x}\right) d_n x = \log u \overset{+}{_n} a.$$

Den Formeln [108] und [111] entsprechend gelten die weiteren Formeln

[130] $$\int_n (u \overset{+}{_n} v \overset{-}{_n} w \text{ etc})\, d_n x = \int_n u \,.\, d_n x \overset{+}{_n} \int_n v \,.\, d_n x \overset{-}{_n} \int_n w \,.\, d_n x \quad \text{etc.},$$

[131] $$\int_n \left(u \overset{+}{_{n+1}} \frac{d_n v}{d_n x}\right) d_n x = u \overset{+}{_{n+1}} v \overset{-}{_n} \int_n \left(v \overset{+}{_{n+1}} \frac{d_n u}{d_n x}\right) d_n x.$$

37. *Beziehungen zwischen den Anti-ratientiationen verschiedener Operationsstufen.* Man könnte auf dem oben angedeuteten Wege fortfahren und

schrittweise alle weiteren Regeln und Sätze für die Anti-ratientenrechnung entwickeln. Es ist aber dieses keineswegs notwendig, da, wie wir sogleich zeigen werden, die Anti-ratientiationen einer bestimmten Stufe durch die Anti-ratientiationen einer beliebigen anderen Stufe ausgeführt werden können. Da nun die Anti-ratientenrechnung der nullten Stufe, d. h. die Integralrechnung, schon vollständig entwickelt ist, so können die bekannten Regeln und Sätze der Integralrechnung in der Ausführung der Anti-ratientenrechnung zu Hilfe gezogen werden. Es ist hierbei zu bemerken, dass dadurch der Integralrechnung logisch keinen Vorzug zuzuschreiben ist. Wären die Regeln und Sätze der Ratienten und Anti-Ratientenrechnung n-ter Stufe bekannt, so könnte man ebenso gut die Regeln und Sätze der Differential- und Integralrechnung aus jenen ableiten.

Es sei $y = {}_n\!\int F(x)\, d_n x = f(x)$, dann ist nach [91]

$$F(x) = \frac{d_n y}{d_n x} = \left(\frac{d_{n+k} y_k}{d_{n+k} x_k}\right)_{-k} = \left(\frac{d_{n-k} y_{-k}}{d_{n-k} y_{-k}}\right)_k$$

und daher

$$\frac{d_{n+k} y_k}{d_{n+k} x_k} = F_k(x), \quad \frac{d_{n-k} y_{-k}}{d_{n-k} x_{-k}} = F_{-k}(x).$$

Hieraus folgt

$$y_k = {}_{n+k}\!\int F_k(x) \,.\, d_{n+k} x_k, \quad y_{-k} = {}_{n-k}\!\int F_{-k}(x) \,.\, d_{n-k} x_{-k},$$

und, da $y = {}_n\!\int F(x)\, d_n x$, so ist schliesslich

$$[132] \quad {}_n\!\int F(x) \,.\, d_n x = \left[{}_{n+k}\!\int F_k(x) \,.\, d_{n+k} x_k \right]_{-k} = \left[{}_{n-k}\!\int F_{-k}(x) \,.\, d_{n-k} x_{-k} \right] .$$

Damit is nicht nur nachgewiesen, dass die Anti-ratientiationen n-ter Stufe durch Anti-ratientiationen beliebiger höherer oder niederer Stufen ausgeführt werden können, es ist durch [132] zugleich die Vorschrift angegeben, wie bei der Ausführung zu verfahren ist. Man muss $F(x)$ durch $F_k(x)$, bz. $F_{-k}(x)$, ersetzen, x_k, bz. x_{-k}, als die unabhängige Veränderliche einführen, und das Resultat der Anti-ratientiation zur entsprechenden Exponenz erheben.

Ist $n=0$, so geht [132] über in

[133] $$\int F(x)\,.\,dx = {}_k\!\!\int F_k(x)\,.\,d_k x_k = {}_{-k}\!\!\int F_{-k}(x)\,.\,d_{-k} x_{-k}$$

und für $k=n$,

[134] $$_n\!\!\int F(x)\,.\,d_n x = \left[\int F_{-n}(x)\,.\,dx_{-n}\right]_n .$$

In [133] erscheint das Integral als ein Anti-ratient k-ter Stufe, in [134] der Anti-ratient n-ter Stufe als ein Integral ausgedrückt.

38. *Isomorphismus der Analyse in den höheren Operationsstufen.* Nachdem wir im Vorhergehenden die Theorie der höheren Operationsstufen im Wesentlichen entwickelt haben, wollen wir hier einen Rückblick auf die gewonnenen Resultate werfen.

(a) Der Versuch einer Erweiterung der Zahlenverhältnisse $a-b$, a/b, führte uns zu den allgemeinen Zahlenverhältnissen $a \overset{-}{_n} b = (a_{-n} - b_{-n})_n$.

(b) Die Zahlenverhältnisse $a \overset{-}{_n} b$ als Operationen zwischen a und b betrachtet führte uns zu den inversen Operationen $a \overset{+}{_n} b = (a_{-n} + b_{-n})_n$.

(c) Es wurde nachgewiesen, dass die Operationen $\underset{n}{\pm}\ \underset{n+1}{\pm}$ genau denselben Gesetzen unterworfen sind wie die Operation $\pm$, $\overset{\times}{\div}$.

(d) Der Zahlenreihe 1, 2, 3,······und der Zahlenschicht N der nullten Operationsstufe entsprechen die Zahlenreihe 1_n, 2_n, 3_n,······und die Zahlenschicht N_n in der n-ten Operationsstufe und die Verknüpfungsgesetze dieser Zahlen in der n-ten Operationsstufe sind mit denen der null-ter Operationsstufe identisch.

(e) Dem Differentiationsprozess der null-ten Operationsstufe entspricht der mit dem Namen Ratientiation bezeichnete Grenzprozess der n-ten Operationsstufe.

(f) Dem Integrationsprozess der null-ten Operationsstufe entspricht der mit dem Namen Anti-ratientiation bezeichnete Prozess der n-ten Operationsstufe.

(g) Die Grundformeln der Differential- und Integralrechnung behalten für die Ratienten- und Anti-ratientenrechnung ihre Gültigkeit, wenn man alle darin vorkommende Operationen und Zahlen durch die entsprechenden Operationen und Zahlen n-ten Stufe ersetzt.

Die Zusammenstellung dieser Resultate berechtigt die wichtige Folgerung.

Jeder Satz der gewöhnlichen Differential- und Integralrechnung führt auf eine unendliche Anzahl anderer Sätze, die man dadurch erhält, dass man

jeden Koeffizienten	N	durch	N_n,
jede Addition	$+$	„	$\overset{+}{n}$,
jede Subtraktion	$-$	„	$\overset{-}{n}$,
jede Multiplikation	$\times$	„	$\overset{+}{n+1}$,
jede Division	$\div$	„	$\overset{-}{n+1}$,
jede Potenz	a^N	„	$a \overset{+}{\scriptstyle n+2} {}^{N}{}_1$,
jede Ableitung	$\dfrac{d^k y}{dx^k}$	„	$\dfrac{d_n^k y}{d_n x^k}$,
jedes Integral	$\int y\,dx$	„	${}_n\!\int y\,d_n x$ ersetzt([1]).

39. *Erläuterungen.* Wir wollen diese Abhandlung zum Abschluss bringen, indem wir den oben angeführten Satz durch einige Beispiele erläutern.

Als erstes Beispiel betrachten wir die Erweiterung des Euler'schen Satzes über die Ableitung einer homogenen Funktion mehrerer Variabeln. Der Satz lautet: Es sei

$$u = \Sigma a x^p y^q z^r \cdots\cdots$$

eine homogene Funktion m-ten Grades in den Veränderlichen $x, y, z, \cdots\cdots$, also

$$p + q + r + \cdots\cdots = m,$$

dann ist

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \cdots\cdots = mu.$$

Es bedeute

$${}_n\Sigma(\ ) = (\ ) \overset{+}{n} (\ ) \overset{+}{n} (\ ) \overset{+}{n} \cdots\cdots$$

also

$${}_0\Sigma(\ ) = (\ ) + (\ ) + (\ ) + \cdots\cdots,\quad {}_1\Sigma(\ ) = (\ ) \times (\ ) \times (\ ) \cdots\cdots,$$
$${}_{-1}\Sigma(\ ) = \log\,[e^{(\ )} + e^{(\ )} + e^{(\ )} \cdots\cdots].$$

Durch Ausführung der obigen Regeln erhält man dann folgende Erweiterung des Euler'schen Satzes:

Es sei

$$u = {}_n\Sigma\,[a \overset{+}{n+1} (x \overset{+}{n+2} p_{n+1}) \overset{+}{n+1} (y \overset{+}{n+2} q_{n+1}) \overset{+}{n+1} (z \overset{+}{n+2} r_{n+1}) \cdots\cdots$$

$$p_n \overset{+}{n} q_n \overset{+}{n} r_n \overset{+}{n} \cdots\cdots = m,$$

(1) Dieses ist die vollständige Erweiterung eines Satzes der für gewisse algebraische Operationen schon von DeMorgan (Trigonometry and Double Algebra, p. 166) angedeutet wurde.

dann ist

$$_n\Sigma\left(x\underset{n+1}{+}\frac{\partial_n u}{\partial_n x}\right)=m\underset{n+1}{+}u.$$

Für $n=1$, und durch gewöhnliche Zeichen ausgedrückt hat man hieraus für

$$u=\Pi a^{(\log x)^p(\log y)^q(\log z)^r\cdots\cdots},\ e^{p+q+r\cdots\cdots}=m,$$

ist

$$\Pi x^{\frac{x}{u}\frac{\partial u}{\partial x}}=u^{\log m}=u^{p+q+r\cdots\cdots}.$$

Für $n=-1$, lautet der Satz: Ist

$$u=\log\Sigma\, e^{a+px+qy+rz+\cdots\cdots},\ \log(p+q+r+\cdots\cdots)=m,$$

so ist

$$\log\Sigma\, e^{u+\log\frac{\partial u}{\partial x}}=m+u,$$

und daher

$$\log\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}+\cdots\cdots\right)=m.$$

Als zweites Beispiel wollen wir die Erweiterung des Taylor'sche Satzes anführen. Schreiben wir den Satz in der Form

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots+\frac{h^k}{k!}f^{(k)}(x)+\frac{h^{k+1}}{(k+1)!}f^{(k+1)}(x+\theta h),$$
$$0<\theta<1,$$

und setzen noch $\frac{d_n^k f}{(d_n x)^k}=f_n^{(k)}(x)$, so ergibt sich, durch Ausführung der obigen Regeln, als Erweiterung des Taylor'schen Satzes

$$f(x\underset{n}{+}h)=f(x)\underset{n}{+}[h\underset{n+1}{+}f_n'(x)]\underset{n}{+}[h\underset{n+2}{+}2_{n+1}\underset{n+1}{+}f_n''(x)\underset{n+1}{-}2_n]\underset{n}{+}\cdots\cdots$$
$$\underset{n}{+}[h\underset{n+1}{+}k_{n+1}\underset{n+1}{+}f_n^{(k)}(x)\underset{n+1}{-}(k\,!)_n]$$
$$\underset{n}{+}[h\underset{n+2}{+}{}^{(k+1)}{}_{n+1}\underset{n+1}{+}f_n^{(k+1)}(x\underset{n}{+}h\underset{n+1}{+}\theta_n)\underset{n+1}{-}{}^{(\overline{k+1\,!})}{}_n],\ 0_n<\theta_n<1_n.$$

Für den speziellen Fall $n=1$, und durch gewöhnliche Zeichen ausgedrückt, hat man ohne Weiteres den schönen schon von Carpenter[1]

[1] American Journal of Mathematics, vol. 35 (1913), p. 105. Der Nachweis bei Carpenter erfordert sieben quarto Seiten.

bewiesenen Satz über die Entwickelung einer Funktion durch ein Produckt einer unendlichen Zahl von Factoren,

$$f(xh) = e^{\log f(x)} e^{\log h \frac{d \log f(x)}{d \log x}} e^{\frac{(\log h)^2}{2!} \frac{d^2 \log f(x)}{(d \log x)^2}} \cdots\cdots e^{\frac{(\log h)^k}{k!} \frac{d^k \log f(x)}{(d \log x)^k}} e,$$

$$R = \frac{(\log h)^{k+1}}{(k+1)!} \frac{d^{k+1} \log f(x.\theta^h)}{(d \log x)^{k+1}}, \; 0 < \theta < 1.$$

Vertauscht man noch x mit h und setzt dann $h=1$, so erhält man das der MacLaurin'sche Reihe analoge Produkt

$$f(x) = e^{\log f(1)} e^{\log x \frac{d \log f(1)}{d \log 1}} e^{\frac{(\log x)}{2!} \frac{d^2 \log f(1)}{(d \log 1)^2}} \cdots\cdots e^{\frac{(\log x)^k}{k!} \frac{d^k \log f(1)}{(d \log 1)^k}} e^R,$$

$$R = \frac{(\log x)^{k+1}}{(k+1)!} \frac{d^{k+1} \log f(\theta^x)}{(d \log 1)^{k+1}}, \; 0 < \theta < 1.$$

Axiomatic Investigation on Number-Systems, II,[1]

by

KUNIZO YONEYAMA, Fukuoka.

(C) Rational Numbers.

Introduction of New Numbers.

Consider the class of integers already defined, and take any two numbers of them, and denote them by α, β. *From this pair of numbers, we construct a new thing which we denote by* (α, β). In order that we may treat it as a number, we lay down the following definitions and postulates.

Definition (A). The two numbers $(\alpha, \beta)_{\beta \neq 0}$ *and* $(\alpha', \beta')_{\beta' \neq 0}$ *are said to be equal when and only when* $\alpha \otimes \beta' = \beta \otimes \alpha'$.

Definition (B). The number $(\alpha, \beta)_{\beta \neq 0}$ *is said to be greater than or less than the number* $(\alpha', \beta')_{\beta' \neq 0}$ *according as* $\alpha \otimes \beta'$ *is greater than or less than* $\beta \otimes \alpha'$.

Postulate (I). $(\alpha, \beta)_{\beta \neq 0} \oplus (\alpha', \beta')_{\beta' \neq 0} = \{(\alpha \otimes \beta') \oplus (\beta \otimes \alpha'), \beta \otimes \beta'\}$.

Postulate (II). $(\alpha, \beta)_{\beta \neq 0} \otimes (\alpha', \beta')_{\beta' \neq 0} = (\alpha \otimes \alpha', \beta \otimes \beta')$.

The assemblage of all numbers thus constructed with the above definitions and postulates is called the system of rational numbers.

Remark. In this new number $(\alpha, \beta)_{\beta \neq 0}$, β is always different from zero. But, for the sake of simplicity, hereafter we shall denote our new number by (α, β) simply and omit $\beta \neq 0$. Therefore, by the symbol (α, β), $\beta \neq 0$ is always understood.

Here we shall study what numbers are defined by the above postulates and definitions.

Theorem 1. The new system of numbers $\{(\alpha, \beta)\}$ *is a more extended one than the system of integers and contains the latter system as its subclass.*

Proof. (1). First of all, when $\alpha = \beta \otimes \gamma$, the new number (α, β) is equivalent to the integer $\alpha \ominus \beta = \gamma$.

For, (a). when $\alpha = \beta \otimes \gamma$, if the relation $(\alpha, \beta) = (\alpha', \beta')$ holds good, then by the definition of equality, we have the relation $\alpha \otimes \beta' = \beta \otimes \alpha'$, and accordingly we have the relation

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[1] Continuation of the paper having the same title in this Journal, vol. 22, pp. 99–137.

$$\beta \otimes \gamma \otimes \beta' = \beta \otimes \alpha',$$

or
$$\gamma \otimes \beta' = \alpha',$$

or
$$\beta' \otimes \gamma = \alpha',$$

or
$$\alpha' \oplus \beta' = \gamma.$$

Therefore
$$\alpha \oplus \beta = \alpha' \oplus \beta'.$$

Thus if the relation $(\alpha, \beta) = (\alpha', \beta')$ holds good, then the relation $\alpha \oplus \beta = \alpha' \oplus \beta'$ also holds good. The converse is also true.

(b). Similarly it may be proved that, when $\alpha = \beta \otimes \gamma$ and $\alpha' = \beta' \otimes \gamma'$, if the relation $(\alpha, \beta) \gtreqless (\alpha', \beta')$ holds good, then the relation $(\alpha \oplus \beta) \gtreqless (\alpha' \oplus \beta')$ or $\gamma \gtreqless \gamma'$ also holds good; and conversely, if the relation $(\alpha \oplus \beta) \gtreqless (\alpha' \oplus \beta')$ holds good, then the relation $(\alpha, \beta) \gtreqless (\alpha', \beta')$ also holds good.

(c). Further, if we put $(\alpha \oplus \beta)$ instead of (α, β) in Postulates (I) and (II), then these postulates are all satisfied.

Thus all definitions and postulates which completely define the new numbers are all satisfied by the integer $(\alpha \oplus \beta)$ when $\alpha = \beta \otimes \gamma$. Therefore, our new number (α, β) may be considered to represent the integer γ when $\alpha = \beta \otimes \gamma$. Moreover, it may easily be seen that any integer may be denoted by one of our new numbers, which has the form (α, β), $(\alpha = (\beta \otimes \gamma))$. Hence we may say that the class of all integers may be represented by the class of new numbers $\{(\alpha, \beta)_{\alpha = (\beta \otimes \gamma)}\}$.

(2). Secondly, when $\alpha \neq \{\beta \otimes \gamma\}$, the number (α, β) has no corresponding one in the class of integers; so the assemblage of them defines a quite new system of numbers. We shall call this class of numbers the class of fractions. Thus the assemblage of all integers and fractions constitutes the class of rational numbers.

Fundamental Properties of the New System of Numbers.

From the above postulates and definitions, it may be proved that all fundamental theorems concerning the operations and comparison of numbers are also true in this new system of numbers. This may be done as in the case of natural numbers and integers.

Removal of Restriction of Operations.

Theorem 2. Two direct operations $\oplus$ *and* $\otimes$ *and an inverse operation* $\ominus$ *can be performed without any restriction in this system of numbers.*

This may be proved in a similar manner as in the case of integers.

In the system of natural numbers and that of integers, the second inverse operation $\alpha \oplus \beta$ is possible only when $\alpha = \beta \otimes \gamma$; in all other cases,

it is impossible. But in the system of rational numbers, the operation $\ominus$ is always possible for any two numbers of the system (except the division by zero).

Theorem 3. The second inverse operation $\ominus$ can be performed without any restriction in this system of numbers (except the division by zero).

Proof. Taking any two numbers (α, β) and (α', β'), we have only to find the numbers (μ, ν) satisfying the relation

$$(\alpha', \beta') \otimes (\mu, \nu) = (\alpha, \beta) \qquad ((\alpha', \beta') \neq 0)$$

to prove this theorem. Now, by Postulate (II), we have the relation

$$(\alpha', \beta') \otimes (\mu, \nu) = (\alpha' \otimes \mu, \beta' \otimes \nu).$$

Therefore, we have to find the integers μ, ν satisfying the relation

$$(\alpha' \otimes \mu, \beta' \otimes \nu) = (\alpha, \beta),$$

or the relation $\qquad \alpha' \otimes \mu \otimes \beta = \beta' \otimes \nu \otimes \alpha \quad$ (by definition of equality),

or the relation $\qquad \mu \otimes (\alpha' \otimes \beta) = \nu \otimes (\beta' \otimes \alpha).$

But to satisfy this condition, it is sufficient to take (μ, ν), such that

$$(\mu, \nu) = (\beta' \otimes \alpha, \alpha' \otimes \beta).$$

Now $\beta' \otimes \alpha$ and $\alpha' \otimes \beta$ are both integers, and since (α', β') is not zero, α' is not zero; and moreover, by the fundamental convention of our number, also β is not zero. Therefore $\alpha' \otimes \beta$ is not zero, so that (μ, ν) is a number belonging to our system of numbers.

Existence of Particular Elements.

Theorem 4. There exists one and only one positive unit-element, namely the number (α, α). Also there exists one and only one negative unit-element, namely the number $(-\alpha, \alpha)$. Moreover, there exists one and only one zero-element, namely the number $(0, \alpha)$.

Definition. The number $(1, \alpha)$ is called a subunit of rational number.

From this definition, we may deduce the theorem.

Theorem 5. Any rational number may be produced from an integer by multiplying it to subunit.

Non-Contradiction and Independence of Postulates.

That Definitions (A), (B), and Postulates (I), (II) in this system of numbers do not imply a contradiction may be seen by a similar consideration as in the case of integers, since all these definitions and postulates are satisfied by the system of integers. Moreover, that Postulates (I) and

(II) are independent of each other may also be seen by the same consideration as in the case of integers.

Denumerable Dense Set of Things.

Denumerable dense set of things are characterised by the following properties:

(1) if a and b are elements of the class K, and $a<b$, then there is at least one element x in K, such that $a<x$ and $x<b$;

(2) the class K is denumerable.

Theorem 6. Our system of numbers is a denumerable dense class of things.

Proof. Take any two numbers of our system (α, β) and (α', β'), and suppose that (α, β) is greater than (α', β'), then by the definition of equality, we have

$$(\alpha, \beta)=(2\,\alpha\beta',\ 2\,\beta\beta'),$$
$$(\alpha', \beta')=(2\,\alpha'\beta,\ 2\,\beta'\beta),$$

and, since (α, β) is greater than (α', β'), we have

$$(2\,\alpha\beta')\,(2\,\beta'\beta) > (2\,\alpha'\beta)\,(2\,\beta\beta'),$$

or

$$2\,\alpha\beta' > 2\,\alpha'\beta.$$

Now $\alpha\beta'$ and $\alpha'\beta$ are integers, and therefore they differ by 1 at least, and therefore $2\,\alpha\beta'$ and $2\,\alpha'\beta$ differ by 2 at least. Accordingly, there is at least one integer lying between $2\,\alpha\beta'$ and $2\,\alpha'\beta$, denote it by γ. Then $(\gamma,\ 2\,\beta\beta')$ satisfies the relation

$$(\alpha', \beta') < (\gamma, 2\,\beta\beta') < (\alpha, \beta),$$

and clearly this number $(\gamma, 2\,\beta\beta')$ belongs to our system of numbers. Thus our system of numbers has the property (1). That it has also the property (2) may be proved by the same method as that used in proving the property (2) of ordinary rational numbers, taking α and β as numerator and denominator of ordinary rational number.

Thus, from the system of integers, we have constructed a more extended system of numbers which satisfies the four great principles:

1. *principle of permanency of form,*
2. *principle of freedom of direct and inverse operations of the first and the second orders,*
3. *principle of non-contradiction,*
4. *principle of denumerable density.*

(D) Real Numbers.

Dedekind has defined the real numbers by the "cut" of the class of rational numbers. Here we shall use his idea and make suitable change of it to fit for our purpose. For this, we shall first derive a system of numbers from the class of rational numbers before to introduce real numbers.

Derived System of Numbers.

Divide the class of all rational numbers into two groups $\mathfrak{A}$, $\mathfrak{B}$, such that every number of $\mathfrak{A}$ is less than any number of $\mathfrak{B}$, then $\mathfrak{A}$ always contains all negative numbers less than a certain negative number, and $\mathfrak{B}$ always contains all positive numbers greater than a certain positive number. We shall call each of $\mathfrak{A}$, $\mathfrak{B}$ the conjugate group of the other. Now make such divisions of all rational numbers in every possible ways. Of all these groups, first consider the assemblage of all groups belonging to the type $\mathfrak{B}$; then there are three kinds of them. For, if we take a rational number $\mathfrak{b}$ and put all rational numbers greater than $\mathfrak{b}$ into the group $\mathfrak{B}$, and all rational numbers less than $\mathfrak{b}$ into another group $\mathfrak{A}$, and put $\mathfrak{b}$ itself into the group $\mathfrak{B}$, then $\mathfrak{B}$ has the least element $\mathfrak{b}$ in it; and if we put $\mathfrak{b}$ into the group $\mathfrak{A}$, then $\mathfrak{B}$ has no least element in it, (instead of it, $\mathfrak{A}$ has the greatest element in it). But the former $\mathfrak{B}$ and the latter $\mathfrak{B}$ differ by only one element $\mathfrak{b}$. Denote by $\mathfrak{B}_1$ the group belonging to the former type and by $\mathfrak{B}_2$ the group belonging to the latter type. Now there is still the third type of them; for, take a natural number 2, for example, and put all rational numbers whose square ($\mathfrak{a} \otimes \mathfrak{a}$ is called the square of $\mathfrak{a}$) is less than 2 into the group $\mathfrak{A}$, and put all rational numbers whose square is greater than 2 into the group $\mathfrak{B}$. Then, since there is no rational number whose square is equal to 2 (which may be proved in the usual manner), the above groups $\mathfrak{A}$, $\mathfrak{B}$ contains all rational numbers, yet $\mathfrak{A}$ has no greatest element and $\mathfrak{B}$ no least element. Therefore the group $\mathfrak{B}$ of this type differs from the types $\mathfrak{B}_1$ and $\mathfrak{B}_2$; denote such group by $\mathfrak{B}_3$. Thus the three kinds of groups of the type $\mathfrak{B}$ are

1 that which has the least element in it, ($\mathfrak{B}_1$ type)

2 that which has no least element in it, but has always a corresponding group $\mathfrak{B}_1$, differing from it by only one element, ($\mathfrak{B}_2$ type)

3 that which has no least element in it, and also has no group $\mathfrak{B}_1$, differing from it by only one element. ($\mathfrak{B}_3$ type)

Now we consider each of these groups as one thing and set up the following definitions and postulates in order to treat it as a number.

Definition (A). Two groups $\mathfrak{B}_m$, $\mathfrak{B}_n$ are said to be equal to each other when and only when the elements of the two groups are identical with or differ from each other by only one element.

Therefore, in order that $\mathfrak{B}_m$ and $\mathfrak{B}_n$ may be equal to each other, both of them must belong to the same kind, or one of them to $\mathfrak{B}_1$ while the other to $\mathfrak{B}_2$, but never one to $\mathfrak{B}_3$ and other to $\mathfrak{B}_1$ or $\mathfrak{B}_2$.

Definition (B). When $\mathfrak{B}_m$ and $\mathfrak{B}_n$ are not equal to each other, from the fundamental property of the groups, one of them wholly contains the other as its proper part. In this case, if $\mathfrak{B}_m$ contains $\mathfrak{B}_n$ as its proper part, then $\mathfrak{B}_m$ is said to be less than $\mathfrak{B}_n$, and conversely $\mathfrak{B}_n$ is said to be greater than $\mathfrak{B}_m$.

Postulate (I). By $\mathfrak{B}_m \oplus \mathfrak{B}_n$ is meant the group which consists of all the numbers resulting from the sums of any element of $\mathfrak{B}_m$ and any element of $\mathfrak{B}_n$. In symbol, if $\mathfrak{B}_m \equiv \{\mathfrak{b}_m\}$ and $\mathfrak{B}_n \equiv \{\mathfrak{b}_n\}$, then $\mathfrak{B}_m \oplus \mathfrak{B}_n \equiv \{\mathfrak{b}_m \oplus \mathfrak{b}_n\}$.

Of the groups belonging to the type $\mathfrak{B}$, that which contains no negative element is called the group of the first kind; and that which contains some negative elements is called the group of the second kind.

Postulate (II). 1. In the case in which both of $\mathfrak{B}_m$ and $\mathfrak{B}_n$ belong to the first kind, by $\mathfrak{B}_m \otimes \mathfrak{B}_n$ is meant the group which consists of all numbers resulting from the products of any element of $\mathfrak{B}_m$ and any element of $\mathfrak{B}_n$. In symbol, $\mathfrak{B}_m \otimes \mathfrak{B}_n \equiv \{\mathfrak{b}_m \otimes \mathfrak{b}_n\}$. 2. In the case in which both of $\mathfrak{B}_m$ and $\mathfrak{B}_n$ belong to the second kind, by $\mathfrak{B}_m \otimes \mathfrak{B}_n$ is meant the group consisting of all numbers resulting from the products of any element of the conjugate group of $\mathfrak{B}_m$ and any element of conjugate group of $\mathfrak{B}_n$. In symbol, $\mathfrak{B}_m \otimes \mathfrak{B}_n \equiv \{\mathfrak{b}'_m \otimes \mathfrak{b}'_n\}$. (Here $\mathfrak{b}'_m$ denotes any element of the conjugate group of $\mathfrak{B}_m$). 3. In the case in which one of $\mathfrak{B}_m$ and $\mathfrak{B}_n$ belongs to the first kind while the other belongs to the second kind, by $\mathfrak{B}_m \otimes \mathfrak{B}_n$ is meant the conjugate of the group consisting of all numbers resulting from the products of any element of $\mathfrak{B}_m$ and any element of the conjugate group of $\mathfrak{B}_n$. In symbol, $\mathfrak{B}_m \otimes \mathfrak{B}_n \equiv \{\mathfrak{b}_m \otimes \mathfrak{b}'_n\}'$. (Here it is supposed that $\mathfrak{B}_m$ belongs to the first kind and $\mathfrak{B}_n$ to the second kind).

Remark. If, in $\{\mathfrak{b}_m \oplus \mathfrak{b}_n\}$, there are elements which are equal to one another, then we make a convention that one of them is reserved and all others are taken away from the group $\{\mathfrak{b}_m \oplus \mathfrak{b}_n\}$, so that there is no repeated element in it. The same convention is made of $\{\mathfrak{b}_m \otimes \mathfrak{b}_n\}$.

The system consisting of all numbers belonging to the type $\mathfrak{B}$ is called a derived number-system of the first kind.

Secondly, consider the assemblage of all groups belonging to the type

$\mathfrak{A}$, then, as in that of the type $\mathfrak{B}$, there are three kinds of them, denote them by $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$. To treat these groups as numbers, we lay down the same definitions and postulates with regard to the equality and the operation $\oplus$ as in the type $\mathfrak{B}$. To get the definition of inequality, we only interchange "less" and "greater" in the definition of the type $\mathfrak{B}$, all others remaining the same. Postulate concerning the operation $\otimes$ is given as follows.

Denote the conjugate groups of $\mathfrak{A}_m$ and $\mathfrak{A}_n$ by $\mathfrak{B}_m$ and $\mathfrak{B}_n$ respectively, then $\mathfrak{B}_m \otimes \mathfrak{B}_n$ is completely determined by Postulate (II), and its result is also a group of the type $\mathfrak{B}$; denote it by $\mathfrak{B}'$. Take the conjugate group of $\mathfrak{B}'$ and denote it by $\mathfrak{A}'$. Then by $\mathfrak{A}_m \otimes \mathfrak{A}_n$ is meant the number $\mathfrak{A}'$.

The system consisting of all numbers belonging to the type $\mathfrak{A}$ is called a derived number-system of the second kind.

From the above postulates we have the following theorems at once.

Theorem. The sum or the product of any two numbers of the type $\mathfrak{A}$ (or the type $\mathfrak{B}$) is also a number of the same type.

Theorem. If $\mathfrak{A}_m$ and $\mathfrak{A}_n$ are conjugate numbers of $\mathfrak{B}_m$ and $\mathfrak{B}_n$ respectively, then $\mathfrak{A}_m \otimes \mathfrak{A}_n$ is also a conjugate number of $\mathfrak{B}_m \otimes \mathfrak{B}_n$.

Theorem. In the above case, $\mathfrak{A}_m \oplus \mathfrak{A}_n$ is also a conjugate number of $\mathfrak{B}_m \oplus \mathfrak{B}_n$ in general; only in the special case, it may happen that $(\mathfrak{A}_m \oplus \mathfrak{A}_n)$ and $(\mathfrak{B}_m \oplus \mathfrak{B}_n)$ lack only one rational number among their elements.

Real Numbers.

With pairs of numbers taken from the above derived number-system, we construct a new system of numbers in the following manner.

From the derived number-systems of the first and second kinds, take two numbers $\mathfrak{A}$, $\mathfrak{B}$ respectively, such that, in $(\mathfrak{A}, \mathfrak{B})$, all rational numbers (with the exception of at most one of them) are contained under one of the following conditions:

(i) none of them is repeated nor lacked,

(ii) one and only one of them is repeated,

(iii) one and only one of them is lacked.

That such three cases may really occur may be seen at once; namely if we donote by $\mathfrak{A}_0$ the group containing all rational numbers less than a rational number $\mathfrak{m}$, but not $\mathfrak{m}$; and by $\mathfrak{A}_1$ that containing $\mathfrak{m}$ also; and by $\mathfrak{B}_0$ that containing all rational numbers greater than $\mathfrak{m}$, but not $\mathfrak{m}$ itself; and $\mathfrak{B}_1$ that containing $\mathfrak{m}$ also; then $\mathfrak{B}_1$ and $\mathfrak{A}_0$ or $\mathfrak{B}_0$ and $\mathfrak{A}_1$ contain all rational numbers in them, every rational numbers occurring only once; $\mathfrak{B}_1$ and $\mathfrak{A}_1$ contain all rational numbers, $\mathfrak{m}$ and only $\mathfrak{m}$ being repeated;

$\mathfrak{B}_0$ and $\mathfrak{A}_0$ contain all rational numbers, $\mathfrak{m}$ and only $\mathfrak{m}$ being lacked.

With such pair of numbers $\mathfrak{A}$, $\mathfrak{B}$, we construct a new thing and denote it by $(\mathfrak{A}, \mathfrak{B})$. In order that we may treat it as a number, we lay down the following definitions and postulates.

Definition (A). The two numbers $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}', \mathfrak{B}')$ are said to be equal to each other when and only when $\mathfrak{A}=\mathfrak{A}'$ and $\mathfrak{B}=\mathfrak{B}'$.

Definition (B). The number $(\mathfrak{A}, \mathfrak{B})$ is said to be greater than or less than the number $(\mathfrak{A}', \mathfrak{B}')$ according as $\mathfrak{A}$ is greater than or less than $\mathfrak{A}'$ (or according as $\mathfrak{B}$ is greater than or less than $\mathfrak{B}'$).

Postulate (I). $(\mathfrak{A}, \mathfrak{B})\oplus(\mathfrak{A}', \mathfrak{B}')=(\mathfrak{A}\oplus\mathfrak{A}'\ \mathfrak{B}\oplus\mathfrak{B}')$.

Postulate (II). $(\mathfrak{A}, \mathfrak{B})\otimes(\mathfrak{A}', \mathfrak{B}')=(\mathfrak{A}\otimes\mathfrak{A}', \mathfrak{B}\otimes\mathfrak{B}')$.

The system of all numbers thus constructed with the above definitions and postulates is called the system of real numbers.

Here we shall study what numbers are defined by the above definitions and postulates.

Theorem 1. The new system of numbers is a more extended one than the system of rational numbers and contains the latter system as its subclass.

Proof. From the mode of construction of new numbers, they may be divided into the following classes.

(i). $\mathfrak{A}$ has the greatest element and $\mathfrak{B}$ has the least element. In this case, these grestest and least elements must be the same rational number as may easily be shown from the property of the new number; denote this element by $\mathfrak{m}$. The case is nothing else than that in which $(\mathfrak{A}, \mathfrak{B})$ has one and only one repeated element.

(ii). $\mathfrak{A}$ has the greatest element and $\mathfrak{B}$ has no least element, or $\mathfrak{B}$ has the least element and $\mathfrak{A}$ has no greatest element; denote this greatest or least element by $\mathfrak{m}$. This case is the one in which $(\mathfrak{A}, \mathfrak{B})$ contains all rational numbers, each occurring only once.

(iii). $\mathfrak{A}$ has no greatest element and $\mathfrak{B}$ has no least element. This case is subdivided into the following two:

(a) $(\mathfrak{A}, \mathfrak{B})$ contains all rational numbers except one and only one rational number, denote this excepted rational number by $\mathfrak{m}$;

(b) $(\mathfrak{A}, \mathfrak{B})$ contains all rational numbers, each occurring only once.

That these two cases (a), (b) may really occur is easily seen from the previous discussion in which we treat the three kinds of the type $\mathfrak{B}$. Also that the case in which $(\mathfrak{A}, \mathfrak{B})$ contains a repeated rational number cannot occur in the case (iii) is easily seen from the definition of $(\mathfrak{A}, \mathfrak{B})$.

Now, in the assemblage of all our numbers, consider the subclass consisting of all the numbers belonging to the cases (i), (ii), $\text{(iii)}_{(a)}$, then

this class of numbers is equivalent to the class of all rational numbers. For, in all three cases, $(\mathfrak{A}, \mathfrak{B})$ has one and only one characteristic rational number, that is, each of $(\mathfrak{A}, \mathfrak{B})$ has one and only one repeated rational number; or one and only one lacked rational number; or one and only one of $\mathfrak{A}$, $\mathfrak{B}$ has the greatest or the least rational number. Denote this characteristic rational number by $\mathfrak{m}$, then $(\mathfrak{A}, \mathfrak{B})$ always determines one and only one rational number $\mathfrak{m}$. Let this rational number $\mathfrak{m}$ correspond to the new number $(\mathfrak{A}, \mathfrak{B})$, then we may at once prove the following relations between these two numbers $\mathfrak{m}$ and $(\mathfrak{A}, \mathfrak{B})$.

(i). When the rational numbers $\mathfrak{m}$ and $\mathfrak{m}'$ correspond to the new numbers $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}', \mathfrak{B}')$, if $(\mathfrak{A}, \mathfrak{B})$ is equal to $(\mathfrak{A}', \mathfrak{B}')$, then $\mathfrak{m}$ is equal to $\mathfrak{m}'$; and conversely.

(ii). If $(\mathfrak{A}, \mathfrak{B})$ is greater than or less than $(\mathfrak{A}', \mathfrak{B}')$, then $\mathfrak{m}$ is also greater than or less than $\mathfrak{m}'$; and conversely.

(iii). If we take $\mathfrak{m}$ and $\mathfrak{m}'$ instead of $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}', \mathfrak{B}')$, they satisfy Postulates (I) and (II).

Thus all definitions and postulates which completely define the new number $(\mathfrak{A}, \mathfrak{B})$ are all satisfied by the rational numbers $\mathfrak{m}$ in the cases (i), (ii), $(\text{iii})_{(a)}$. Therefore our new numbers in these cases may be considered to represent the rational number $\mathfrak{m}$. Moreover, it may easily be seen that any rational number $\mathfrak{m}$ may be denoted by one of our new numbers. Hence we may say that the class of all rational numbers may be represented by our new numbers in the cases (i), (ii), $(\text{iii})_{(a)}$.

Next consider the assemblage of all new numbers belonging to the case $(\text{iii})_{(b)}$. The number of this class has no corresponding element in the class of rational numbers by the above mode of consideration. Therefore, the assemblage of them defines a quite new class of numbers; we shall call it the class of irrational numbers. Thus the assemblage of all rational and irrational numbers constitutes the system of real numbers.

Fundamental Properties of the New System of Numbers.

Theorem 2. If $(\mathfrak{A}, \mathfrak{B})$ *and* $(\mathfrak{A}', \mathfrak{B}')$ *are any two numbers of our system, then* $(\mathfrak{A}, \mathfrak{B}) \oplus (\mathfrak{A}', \mathfrak{B}')$ *and* $(\mathfrak{A}, \mathfrak{B}) \otimes (\mathfrak{A}', \mathfrak{B}')$ *are also elements of our system.*

Proof. (i). If $\mathfrak{A}$ and $\mathfrak{A}'$ have the greatest elements $\mathfrak{a}$ and $\mathfrak{a}'$, and $\mathfrak{B}$ and $\mathfrak{B}'$ have the least elements $\mathfrak{b}$ and $\mathfrak{b}'$, then $\mathfrak{A} \oplus \mathfrak{A}'$ contains all rational numbers less than or equal to $\mathfrak{a} \oplus \mathfrak{a}'$; and $\mathfrak{B} \oplus \mathfrak{B}'$ contains all rational numbers greater than or equal to $\mathfrak{b} \oplus \mathfrak{b}'$. But, in this case, as was already remarked, $\mathfrak{a} \oplus \mathfrak{a}'$ and $\mathfrak{b} \oplus \mathfrak{b}'$ are equal to each other; therefore $\mathfrak{A} \oplus \mathfrak{A}'$ and $\mathfrak{B} \oplus \mathfrak{B}'$ contain all rational numbers and they have one and

only one rational number $\mathfrak{a}\oplus\mathfrak{a}'=\mathfrak{b}\oplus\mathfrak{b}'$ in common, so that $(\mathfrak{A}\oplus\mathfrak{A}', \mathfrak{B}\oplus\mathfrak{B}')$ belongs to our system.

(ii). If $\mathfrak{A}$ and $\mathfrak{A}'$ have the greatest elements $\mathfrak{a}$ and $\mathfrak{a}'$, and at least one of $\mathfrak{B}$ and $\mathfrak{B}'$ have no least element, then $\mathfrak{A}\oplus\mathfrak{A}'$ contains all rational numbers less than or equal to $\mathfrak{a}\oplus\mathfrak{a}'$; and $\mathfrak{B}\oplus\mathfrak{B}'$ contains all rational numbers greater than $\mathfrak{a}\oplus\mathfrak{a}'$, but not $\mathfrak{a}\oplus\mathfrak{a}'$ itself; therefore $\mathfrak{A}\oplus\mathfrak{A}'$ and $\mathfrak{B}\oplus\mathfrak{B}'$ contains all rational numbers, every number occurring only once, so that $(\mathfrak{A}\oplus\mathfrak{A}', \mathfrak{B}\oplus\mathfrak{B}')$ also belongs to our system. The same is true when $\mathfrak{B}$ and $\mathfrak{B}'$ have the least elements and at least one of $\mathfrak{A}$ and $\mathfrak{A}'$ has no greatest element.

(iii). If at most one of $\mathfrak{A}$ and $\mathfrak{A}'$ has the greatest element and at most one of $\mathfrak{B}$ and $\mathfrak{B}'$ has the least element, then both $\mathfrak{A}\oplus\mathfrak{A}'$ and $\mathfrak{B}\oplus\mathfrak{B}'$ have neither the greatest nor the least element. If $\mathfrak{a}$ and $\mathfrak{a}'$ denote any two elements of $\mathfrak{A}$ and $\mathfrak{A}'$ respectively, then $\mathfrak{A}\oplus\mathfrak{A}'$ contains all rational numbers less than or equal to $\mathfrak{a}\oplus\mathfrak{a}'$; and if $\mathfrak{b}$ and $\mathfrak{b}'$ denote any two elements of $\mathfrak{B}$ and $\mathfrak{B}'$, then $\mathfrak{B}\oplus\mathfrak{B}'$ contains all rational numbers greater than or equal to $\mathfrak{b}\oplus\mathfrak{b}'$. Now we can make the difference of $\mathfrak{a}\oplus\mathfrak{a}'$ and $\mathfrak{b}\oplus\mathfrak{b}'$ as small as we please by taking $\mathfrak{a}$, $\mathfrak{a}'$, $\mathfrak{b}$ and $\mathfrak{b}'$ suitably, since we can make so of the difference of $\mathfrak{a}$ and $\mathfrak{b}$, and also of the difference of $\mathfrak{a}'$ and $\mathfrak{b}'$. Hence we can deduce that $\mathfrak{A}\oplus\mathfrak{A}'$ and $\mathfrak{B}\oplus\mathfrak{B}'$ must contain all rational numbers with the exception of at most one rational number. To prove this, suppose that, if possible, they would not contain two rational numbers $\mathfrak{m}$ and $\mathfrak{n}$ $(\mathfrak{m}>\mathfrak{n})$, then $\mathfrak{B}\oplus\mathfrak{B}'$ could contain no rational number less than $\mathfrak{m}$ and $\mathfrak{A}\oplus\mathfrak{A}'$ could contain no rational number greater than $\mathfrak{n}$. For, if $\mathfrak{B}\oplus\mathfrak{B}'$ contained a rational number $\mathfrak{k}$ $(\mathfrak{k}<\mathfrak{m})$, then, since $\mathfrak{B}\oplus\mathfrak{B}'$ must contain all rational numbers greater than $\mathfrak{k}$, so it would contain $\mathfrak{m}$, contrary to our supposition; and if $\mathfrak{A}\oplus\mathfrak{A}'$ contained a rational number $\mathfrak{k}$ $(\mathfrak{k}>\mathfrak{n})$, then since $\mathfrak{A}\oplus\mathfrak{A}'$ must contain all rational numbers less than $\mathfrak{k}$, so it would contain $\mathfrak{n}$, again contrary to our supposition. Therefore all rational numbers of $\mathfrak{A}\oplus\mathfrak{A}'$ must be less than $\mathfrak{n}$, and all rational numbers of $\mathfrak{B}\oplus\mathfrak{B}'$ must be greater than $\mathfrak{m}$, so that the difference of any element of $\mathfrak{A}\oplus\mathfrak{A}'$ and any element of $\mathfrak{B}\oplus\mathfrak{B}'$ would be greater than $\mathfrak{m}\ominus\mathfrak{n}$, again contrary to the above assertion. Thus it follows that $\mathfrak{A}\oplus\mathfrak{A}'$ and $\mathfrak{B}\oplus\mathfrak{B}'$ contain all rational numbers except at most one rational number. Next $\mathfrak{A}\oplus\mathfrak{A}'$ and $\mathfrak{B}\oplus\mathfrak{B}'$ can have no common rational numbers, since any element of $\mathfrak{A}\oplus\mathfrak{A}'$ is less than any element of $\mathfrak{B}\oplus\mathfrak{B}'$ in this case. Therefore again $(\mathfrak{A}\oplus\mathfrak{A}', \mathfrak{B}\oplus\mathfrak{B}')$ belongs to our system.

Thus, in all cases, $(\mathfrak{A}, \mathfrak{B})\oplus(\mathfrak{A}', \mathfrak{B}')$ belongs to our system of numbers.

Remark. Even when both $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}', \mathfrak{B}')$ denote irrational numbers and accordingly each of them contains all rational numbers, it may happen that their sum $(\mathfrak{A} \oplus \mathfrak{A}', \mathfrak{B} \oplus \mathfrak{B}')$ does not contain a certain rational number. For example, when they denote irrational numbers $m.\, a_1\, a_2 \cdots\cdots$ and $-n.\, a_1\, a_2 \cdots\cdots$ their sum denotes the rational number $m \ominus n$. But this rational number is contained neither in $\mathfrak{A} \oplus \mathfrak{A}'$ nor in $\mathfrak{B} \oplus \mathfrak{B}'$. All other rational numbers are contained either in $\mathfrak{A} \oplus \mathfrak{A}'$ or in $\mathfrak{B} \oplus \mathfrak{B}'$.

Similarly, it may be proved that $(\mathfrak{A}, \mathfrak{B}) \otimes (\mathfrak{A}', \mathfrak{B}')$ belongs also to our system of numbers.

Theorem 3. If $(\mathfrak{A}, \mathfrak{B})$ *denotes an irrational number, then it is greater than any rational number contained in* $\mathfrak{A}$ *and is less than any rational number contained in* $\mathfrak{B}$.

Proof. Take any rational number $\mathfrak{a}$ contained in $\mathfrak{A}$ and denote by $\mathfrak{A}_1$ the class of all rational numbers less than $\mathfrak{a}$, and denote by $\mathfrak{B}_1$ the class of all rational numbers greater than $\mathfrak{a}$, then the real number $(\mathfrak{A}_1, \mathfrak{B}_1)$ denotes the rational number $\mathfrak{a}$ as was already stated. Now since $(\mathfrak{A}, \mathfrak{B})$ is an irrational number, $\mathfrak{A}$ has no greatest element, so that $\mathfrak{a}$ cannot be the greatest element of it. Accordingly $\mathfrak{A}$ contains more than one number greater than $\mathfrak{a}$. Thus, by Definition (A), $\mathfrak{A}$ is not equal to $\mathfrak{A}_1$, and moreover it contains $\mathfrak{A}_1$ as its proper part. Therefore, by Definition (B), $\mathfrak{A}$ is greater than $\mathfrak{A}_1$ and accordingly $(\mathfrak{A}, \mathfrak{B})$ is greater than $(\mathfrak{A}_1, \mathfrak{B}_1)$. Similarly $(\mathfrak{A}, \mathfrak{B})$ is less than any rational number $\mathfrak{b}$ contained in $\mathfrak{B}$. Having proved these fundamental theorems, we can prove all other fundamental theorems by a similar method as that used in the case of Dedekind's "cut."

Removal of Restriction of Operations.

In this system of numbers, the four operations (addition, multiplication, subtraction and division) can be performed without any restriction. Moreover, we may prove that any evolution of any positive number of this class can also be performed without any restriction, though it is not so in all previous systems of numbers. For, to find n^{th} root of $(\mathfrak{A}, \mathfrak{B})$, take any positive rational number $\mathfrak{m}$ and if its n^{th} power is less than $(\mathfrak{A}, \mathfrak{B})$, then put it into the group $\mathfrak{A}_1$; and if its n^{th} power is greater than $(\mathfrak{A}, \mathfrak{B})$, then put it into the group $\mathfrak{B}_1$ and if its n^{th} power is equal to $(\mathfrak{A}, \mathfrak{B})$, then put it into one of $\mathfrak{A}_1$ and $\mathfrak{B}_1$; and do this of all positive rational numbers, then any positive rational number is contained in one and only one of $\mathfrak{A}_1$ and $\mathfrak{B}_1$ and none of $\mathfrak{A}_1$ and $\mathfrak{B}_1$ is vacant. Now add to $\mathfrak{A}_1$ all negative rational numbers and zero, then $(\mathfrak{A}_1, \mathfrak{B}_1)$ contains all rational numbers, every number occurring only once. Therefore $(\mathfrak{A}_1\ \mathfrak{B}_1)$ is a number of our

system. Next raise $(\mathfrak{A}_1, \mathfrak{B}_1)$ to the n^{th} power and denote it by $(\mathfrak{A}_2, \mathfrak{B}_2)$, then it may easily be seen that $\mathfrak{A}_2$ contains all rational numbers (except at most one element) of $\mathfrak{A}$, and $\mathfrak{B}_2$ those of $\mathfrak{B}$, so that $(\mathfrak{A}_2, \mathfrak{B}_2)$ is equal to $(\mathfrak{A}, \mathfrak{B})$. Therefore $(\mathfrak{A}_1, \mathfrak{B}_1)$ is a n^{th} root of $(\mathfrak{A}, \mathfrak{B})$.

Existence of Particular Elements.

Theorem 4. There exists one and only one zero-element in our system of numbers.

Proof. Denote by $\mathfrak{A}_0'$ the class containing all rational numbers less than zero, and by $\mathfrak{A}_0$ the class containing $\mathfrak{A}_0'$ and moreover zero itself. Further, denote by $\mathfrak{B}_0'$ the class containing all rational numbers greater than zero, and by $\mathfrak{B}_0$ the class containing $\mathfrak{B}_0'$ and moreover zero itself. Then by the definition of equality, we have

$$(\mathfrak{A}_0, \mathfrak{B}_0)=(\mathfrak{A}_0' \ \mathfrak{B}_0')=(\mathfrak{A}_0', \mathfrak{B}_0)=(\mathfrak{A}_0, \mathfrak{B}_0'),$$

and moreover, by the postulate concerning $\oplus$, we have

$$(\mathfrak{A}_0, \mathfrak{B}_0)\oplus(\mathfrak{A}_0, \mathfrak{B}_0)=(\mathfrak{A}_0\oplus\mathfrak{A}_0, \mathfrak{B}_0\oplus\mathfrak{B}_0)=(\mathfrak{A}_0, \mathfrak{B}_0).$$

Therefore $(\mathfrak{A}_0, \mathfrak{B}_0)$ is a zero-element; and that there is no other zero-element may be proved as follows.

Suppose that $(\mathfrak{A}, \mathfrak{B})$ is a zero-element, then we have

$$(\mathfrak{A}, \mathfrak{B})\oplus(\mathfrak{A}, \mathfrak{B})=(\mathfrak{A}\oplus\mathfrak{A}, \mathfrak{B}\oplus\mathfrak{B})=(\mathfrak{A}, \mathfrak{B}).$$

Therefore we have

$$\mathfrak{A}\oplus\mathfrak{A}=\mathfrak{A}, \qquad \oplus\mathfrak{B}=\mathfrak{B}.$$

Now if $\mathfrak{A}$ has the greatest element $\mathfrak{a}$, then from the relation $\mathfrak{A}\oplus\mathfrak{A}=\mathfrak{A}$ we must have $\mathfrak{a}\oplus\mathfrak{a}=\mathfrak{a}$, therefore $\mathfrak{a}=0$; accordingly $\mathfrak{A}$ is equal to $\mathfrak{A}_0$, and thus $(\mathfrak{A}, \mathfrak{B})$ is equal to $(\mathfrak{A}_0, \mathfrak{B}_0)$ or $(\mathfrak{A}_0, \mathfrak{B}_0')$. The same may be said of $(\mathfrak{A}, \mathfrak{B})$ when $\mathfrak{B}$ has the least element; namely in this case $(\mathfrak{A}, \mathfrak{B})$ is equal to $(\mathfrak{A}_0, \mathfrak{B}_0)$ or $(\mathfrak{A}_0' \ \mathfrak{B}_0)$. If $\mathfrak{A}$ has no greatest element and $\mathfrak{B}$ has no least element, then denote $(\mathfrak{A}, \mathfrak{B})$ by $\mathfrak{k}$ ($\mathfrak{k}$ may be rational or irrational), then $\mathfrak{k}$ is greater than any rational number contained in $\mathfrak{A}$ and is less than any rational number contained in $\mathfrak{B}$. Now in order that $\mathfrak{A}\oplus\mathfrak{A}=\mathfrak{A}$, $\mathfrak{B}\oplus\mathfrak{B}=\mathfrak{B}$ may hold good, it is necessary that $\mathfrak{k}\oplus\mathfrak{k}=\mathfrak{k}$ must hold good by the definition of equality and the postulate of the operation $\oplus$. Therefore if $\mathfrak{k}$ is rational, then $\mathfrak{k}$ is zero, and $\mathfrak{A}$ and $\mathfrak{B}$ are equal to $\mathfrak{A}_0'$ and $\mathfrak{B}_0'$, so that $(\mathfrak{A}, \mathfrak{B})$ is equal to $(\mathfrak{A}_0', \mathfrak{B}_0')$. Next, if $\mathfrak{k}$ is irrational, then $\mathfrak{k}$ is different from the rational number 0 and is greater than or less than it. Suppose that it is greater than 0, then $\mathfrak{A}$ contains 0 and some rational numbers greater than 0. Now, by the definition of $\mathfrak{A}$ and $\mathfrak{B}$, we can always take

two rational numbers $\mathfrak{a}''$ and $\mathfrak{b}''$ from $\mathfrak{A}$ and $\mathfrak{B}$ respectively, such that $\mathfrak{b}'' \ominus \mathfrak{a}''$ is equal to or less than a rational number $\mathfrak{n}$, however small $\mathfrak{n}$ may be. Therefore all rational numbers of $(\mathfrak{A}, \mathfrak{B})$, lying between $\mathfrak{a}''$ and $\mathfrak{b}''$, is different from $\mathfrak{a}''$ by a number less than $\mathfrak{n}$. If we take $\mathfrak{n}$ sufficiently small, then we can always take $\mathfrak{a}''$, such that it is positive and is greater than $\mathfrak{n}$; and accordingly $\mathfrak{a}'' \oplus \mathfrak{a}''$ is greater than $\mathfrak{a}'' \oplus \mathfrak{n} \geqq \mathfrak{b}''$, and so it must belong to $\mathfrak{B}$. Therefore $\mathfrak{A} \oplus \mathfrak{A}$ would be greater than $\mathfrak{A}$, and so in this case the equality would not hold good. When $\mathfrak{k}$ is less than zero, a similar discussion leads to the same result. Hence it follows that, if $(\mathfrak{A}, \mathfrak{B}) \oplus (\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}, \mathfrak{B})$, then $(\mathfrak{A}, \mathfrak{B})$ cannot be irrational. Q. E. D.

Theorem 5. There exists one and only one positive unit-element in our system of numbers.

Proof. When $\mathfrak{A}_1$, $\mathfrak{A}_1'$, $\mathfrak{B}_1$, $\mathfrak{B}_1'$ denote the numbers of the same nature as in the above theorem, taking 1 instead of 0, we have

$$(\mathfrak{A}_1, \mathfrak{B}_1) = (\mathfrak{A}_1', \mathfrak{B}_1') = (\mathfrak{A}_1', \mathfrak{B}_1) = (\mathfrak{A}_1, \mathfrak{B}_1'),$$

and
$$(\mathfrak{A}_1, \mathfrak{B}_1) \otimes (\mathfrak{A}_1, \mathfrak{B}_1) = (\mathfrak{A}_1 \otimes \mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{B}_1) = (\mathfrak{A}_1, \mathfrak{B}_1).$$

Therefore $(\mathfrak{A}_1, \mathfrak{B}_1)$ is a positive unit-element, and that there is no other positive unit-element may be proved as in the above theorem.

Theorem 6. There exists one and only one negative unit-element in our system of numbers.

This may be treated as in the above theorem.

Further, that the definitions of equality and inequality and the postulates concerning the operations $\oplus$ and $\otimes$ do not imply contradiction in this system of numbers, and moreover, that Postulates (I) and (II) are independent of each other may be seen by the same consideration as in the case of rational numbers.

Continuous Set of Things.

The continuous set of things is characterised by the following properties:

(i) it is a simply ordered class,

(ii) it satisfies Dedekind's postulate,

(iii) it satisfies the postulate of density.

The linear continuous set of things is a continuous set which satisfies a further condition called the postulate of linearity(1).

Theorem 6. Our system of numbers is a linear continuous set of things.

(1) See Huntington, The Continuum, p. 44.

This may be proved by the mode of construction of our numbers and our definitions of equality and inequality.

Thus, from the system of rational numbers, we have constructed a more extended system of numbers which satisfies the four great principles:

1. *principle of permanency of form;*
2. *principle of freedom of direct and inverse operations of the first and second orders,*
3. *principle of non-contradiction,*
4. *principle of linear continuity.*

(E) Complex Numbers.

Consider the class of real numbers already defined, and take any two numbers of them, and denote them by a, b. *From this pair of numbers, we construct a new thing which we denote by* (a, b). In order that we may treat this thing as a number, we lay down the following definitions and postulates.

Definition (A). *The two numbers* (a, b) *and* (a', b') *are said to be equal when and only when* $a=a'$ *and* $b=b'$.

Definition (B). *The number* (a, b) *is said to be greater than or less than the number* (a', b') *according as* a *is greater than or less than* a', *and if* a *is equal to* a', *then according as* b *is greater than or less than* b'.

Postulate (I). $(a, b) \oplus (a', b') = (a \oplus a',\ b \oplus b')$.

Postulate (II). $(a, b) \otimes (a', b') = (aa' \ominus bb',\ ab' \oplus a'b)$.

The assemblage of all numbers thus constructed with the above definitions and postulates is called the system of complex numbers.

Here we shall study what numbers are defined by the above definitions and postulates.

Theorem 1. The new system of numbers is a more extended one than the system of real numbers and contains the latter system as its subclass.

Proof. When $b=0$, the new number (a, b) is equivalent to the real number a. For, if two numbers $(a, b')_{b=0}$ and $(a', b')_{b'=0}$ are equal to each other, then, by Definition (A), a is equal to a'; and conversely. If $(a, b)_{b=0}$ is greater than or less than $(a', b')_{b'=0}$, then, by Definition (B), a is greater than or less than a'; and conversely. Further, if we take a and a' instead of $(a, 0)$ and $(a', 0)$, Postulates (I) and (II) are also satisfied. Moreover, to any real number a, there is always a corresponding new number $(a, 0)$ which is equivalent to it. Therefore, the class of all real numbers may be represented by the class of new numbers $\{(a, b)_{b=0}\}$.

When $b \neq 0$, the class of the new numbers represents a quite new class of numbers. In this case, if $a=0$, then (a, b) is called an imaginary number; and if $a \neq 0$, then (a, b) is called an essential complex number.

Fundamental Properties of the New System of Numbers.

From the above postulates and definitions, it may be proved that all fundamental theorems of number-systems already obtained hold good in this new system of numbers. Since the method of proving them presents no difficulty, we shall not enter into it. Only the following remark will be added as it is one of characteristic properties of this system of numbers.

In any system of numbers previously treated, there is no element, such that its square is equal to a negative number. But, in this new system of numbers, there are such elements. For example, consider the number $(0, b)$, then we have

$$(0, b) \otimes (0, b) = (0 \ominus b^2, 0 \oplus 0) = (-b^2, 0).$$

But the number $(0, 0)$ is a zero-element since the relation $(0, 0) \oplus (0, 0) = (0, 0)$ holds good; and $(0, 0)$ is greater than $(-b^2, 0)$ by Definition (B). Therefore $(-b^2, 0)$ is a negative real number. Now b may be any real number and so $(0, b)$ may be any imaginary number. Thus we have the theorem.

Theorem 2. The square of any imaginary number is equal to a negative real number.

Existence of Particular Elements.

Theorem 3. There exists one and only one zero-element $(0, 0)$ *in our system of numbers.*

Theorem 4. There exist one and only one positive unit-element $(1, 0)$, *and one and only one negative unit-element* $(-1, 0)$ *in our system of numbers.*

As we have already seen, the numbers whose square is equal to a positive unit-element are positive and negative unit-elements and only these. Now in our new system of numbers, besides such numbers, there are numbers whose square is equal to a negative unit-element. For example,

$$(0, 1) \otimes (0, 1) = (-1, 0),$$

$$(0, -1) \otimes (0, -1) = (-1, 0).$$

Namely there are two such numbers, and it may be proved that there is no other number having such a property. For, if (a, b) is any number having such a property, then it must satisfy the relation

$$(a,\ b)\otimes(a,\ b)=(aa\ominus bb,\ ab\oplus ab)=(-1,\ 0)$$

and accordingly the relations $aa\ominus bb=-1$ and $2ab=0$ must hold good. Now b cannot be zero, for, if so, the relation $aa=-1$ would hold good, contrary to the property of real number a. Therefore, from $2ab=0$, it follows that a must be zero and accordingly follows the relation $b^2=1$. Hence b must be $+1$ or -1. Thus we have two and only two numbers $(0,+1)$ and $(0,-1)$ as the required ones. We shall call these numbers positive imaginary unit and negative imaginary unit respectively; and we shall denote them by $+i$ and $-i$ respectively.

Theorem 5. Any imaginary number may be produced from a real number by multiplying it to an imaginary unit.

Proof. Any imaginary number may be represented by $(0,\ b)$ and any real number by $(a,\ 0)$. But, by Postulate (II), we have

$$(0,\ b)=(0,\ 1)\otimes(b,\ 0).$$

Therefore, any imaginary number is represented by the product of the imaginary unit $(0,\ 1)$ and a real number $(b,\ 0)$.

Theorem 6. Any essential complex number may be produced by the sum of a real number and an imaginary number.

For, by Postulate (II), we have

$$(a,\ b)=(a,\ 0)\oplus(0,\ b)=(a,\ 0)\oplus\{(b,\ 0)\otimes(0,\ 1)\}=a\oplus bi.$$

Further, that our postulates and definitions do not imply contradiction, and moreover, that Postulates (I) and (II) are independent of each other may be seen by the same consideration as in the case of real numbers.

Removal of Restriction of Operations.

In the system of natural numbers, three direct operations (addition, multiplication and involution) can be performed without any restriction, but none of their inverse operations (subtraction, division, evolution and logarithmic operation) can be done so. In the system of integers, all direct operations and only one of inverse operations (subtraction) can be performed without any restriction; and in the system of rational numbers, one more inverse operation (division) can be done so. Further, in the system of real numbers, one more inverse operation (evolution of positive numbers) can be done so. Lastly, in the system of complex numbers, all direct and all inverse operations can be performed without any restriction. Thus, for the first time, in the system of complex numbers, we get perfect freedom of operations. Hence the theorem.

Theorem 7. In this system of numbers, all direct and all inverse operations of the first, the second and the third orders can be performed without any restriction.

Moreover, considered as a set of things, the system of complex numbers forms a twofold continuous set.

Thus, from the system of real numbers, we have constructed a more extended system of numbers which satisfies the four great principles:

1. *principle of permanency of form,*
2. *principle of perfect freedom of all operations of the first, the second and the third orders,*
3. *principle of non-contradiction,*
4. *principle of twofold continuity.*

Conclusion. Thus far, by a simple, unified and rigorous method, we have constructed many abstract systems of numbers, and thus have extended the domain of numbers step by step, starting from only one thing and only two operations, and successively introducing new systems of numbers under the same principle. At last we have reached the great mental structure having the following fundamental properties:

1. *it satisfies all fundamental laws of comparison and operations,*
2. *it enjoys perfect freedom of the seven fundamental operations,*
3. *it forms a continuous set of things,*
4. *it is free from contradiction.*

And the method of constructing these structures is characterised by the following two important properties:

1. *it introduces new systems of numbers by one and the same method (method of "pair of numbers"),*
2. *it introduces new systems of numbers without any aid of concrete or geometrical quantities.*

By this method, we may proceed one step further, and may construct quaternion from a pair of complex numbers, though, in this system, the commutative law does not hold good generally as is well known.

(F). **Quaternion.**

Introduction of New Numbers.

Consider the system of complex numbers already defined, and take any two numbers of them, and denote them by α, β. *From this pair of numbers, we construct a new thing which we denote by* (α, β). In order that we may treat it as a number, we lay down the following definitions and postulates.

Definition (*A*). *The two numbers* (α, β) *and* (α', β') *are said to be equal to each other when and only when* $\alpha=\alpha'$ *and* $\beta=\beta'$.

Definition (*B*). *The number* (α, β) *is said to be greater than or less than the number* (α', β') *according as* α *is greater than or less than* α', *and when* α *is equal to* α', *according as* β *is greater than or less than* β'.

Postulate (*I.*) $(\alpha, \beta)\oplus(\alpha', \beta')=(\alpha\oplus\alpha', \beta\oplus\beta')$.

Postulate (*II.*) $(\alpha, \beta)\otimes(\alpha', \beta')=(\alpha\alpha'\ominus\beta\bar{\beta}', \alpha\beta'\oplus\bar{\alpha}'\beta)$,

where $\bar{\gamma}$ *denotes the conjugate of* γ; *namely if* $\gamma=a\oplus bi$, *then* $\bar{\gamma}=a\ominus bi$.

The assemblage of all numbers thus constructed with these definitions and postulates is called the system of quaternions.

Here we shall study what numbers are represented by the above definitions and postulates.

Theorem 1. The new system of numbers is a more extended one than the system of complex numbers and contains it as its subclass.

Proof. When $\beta=0$, the new number (α, β) is equivalent to the complex number α. For, if two numbers $(\alpha, \beta)_{\beta=0}$ and $(\alpha', \beta')_{\beta'=0}$ are equal to each other, then, by Definition (*A*), α is equal to α', and conversely. If $(\alpha, \beta)_{\beta=0}$ is greater than or less than $(\alpha', \beta')_{\beta'=0}$, then, by Definition (*B*), α is greater than or less than α', and conversely. Further, if we take α and α' instead of $(\alpha, \beta)_{\beta=0}$ and $(\alpha', \beta')_{\beta'=0}$, then Postulates (I) and (II) are also satisfied. Moreover, to any complex number α, there is always a corresponding one $(\alpha, \beta)_{\beta=0}$ in our system of numbers, which is equivalent to it. Therefore, the class of all complex numbers may be represented by the class of new numbers $\{(\alpha, \beta)_{\beta=0}\}$; and we may put $(\alpha, \beta)_{\beta=0}=\alpha$.

When $\beta\neq0$, the class of such new numbers represents a quite new class of numbers. Thus the system of complex numbers may be considered as a subclass of quaternions.

Fundamental Properties and New Units.

If we put $\alpha=\alpha'=0$ and $\beta=\beta'=1$ in Postulate (II), then we have

(1) $$(0, 1)^2=(-1, 0)=-1.$$

Therefore, if we put $(0, 1)=j$, then we have $j^2=-1$; in this respect, j resembles very much to the imaginary unit i, but it is not identical with i as will be seen hereafter. We shall take this number as a new unit. Using this new unit j, we may write any number of this system in the form $\alpha\oplus\beta j$. For, by Postulate (II), we have

(2) $$(\beta, 0)\otimes(0, 1)=(0, \beta),$$

and by the above theorem, we have

$$(\alpha,\ 0)=\alpha,$$

and moreover, by Postulate (I), we have

$$(\alpha,\ 0)\oplus(0,\ \beta)=(\alpha,\ \beta).$$

Therefore, we have

$$(3)\qquad (\alpha,\ \beta)=(\alpha,\ 0)\oplus(0,\ \beta)=\alpha\oplus(\beta,\ 0)\ (0,\ 1)=\alpha\oplus\beta j.$$

We shall call this unit *the first quaternion-unit.*

Next if we put $\alpha=\alpha'=0$ and $\beta=\beta'=i$ in Postulate (II), then we have

$$(0,\ i)\otimes(0,\ i)=\{0-i(-i),\ 0\}=(i^2,\ 0)=-1$$

or (4) $$(0,\ i)^2=-1.$$

Therefore, if we put $(0,\ i)=k$, then we have $k^2=-1$; in this respect, k resembles very much the imaginary unit i and also the first quaternion unit j. But it is identical neither with i nor with j as will be seen hereafter. We shall take this number as a new unit. Using this new unit k, we may write any number of this class in the form $\alpha\oplus\beta k$ as in the case of the first quaternion unit j. For, by Postulate (II), we have

$$(-i\beta,\ 0)\otimes(0,\ i)=(0,-i\beta\otimes i\oplus 0)=(0,\ \beta),$$

and by Postulate (I), we have

$$(\alpha,\ 0)\oplus(0,\ \beta)=(\alpha,\ \beta).$$

Therefore, we have

$$(\alpha,\beta)=(\alpha,0)\oplus(0,\beta)=\alpha\oplus\{(-i\beta,0)\otimes(0,i)\}=\alpha\oplus(-i\beta)k=\alpha\oplus\beta' k,$$

(where $\beta'=-i\beta$ is a complex number). We shall call this unit *the second quaternion-unit.*

By using these four units $(1, i, j, k)$, we may express any quaternion as a linear homogeneous function of these units with real numbers as its coefficients. Before to prove this, we have to prove the following theorem:

Theorem 2. In this new system of numbers, there are two and only two numbers whose square is equal to $+1$, but there are an infinity of numbers whose square is equal to -1. In the latter case, there is a very simple relation among a_1, a_2, b_1, b_2 of the number $(\alpha,\ \beta)=(a_1\oplus a_2 i,\ b_1\oplus b_2 i)$, whose square is -1. If we confine a_1, a_2, b_1, b_2 to be integers, there are four and only four kinds of numbers whose square is $+1$ or -1; namely one and only one of a_1, a_2, b_1, b_2 is ±1 while all others are zero. These give us $\pm1, \pm i, \pm j, \pm k$ respectively. They are what we have taken already as units of our number-system.

Proof. From the relation $(\alpha, \beta)^2=(\alpha^2 \ominus \beta\bar{\beta}), \alpha\beta \oplus \bar{\alpha}\beta)$, we have

$$(a_1 \oplus a_2 i, b_1 \oplus b_2 i)^2=\{a_1^2 \ominus a_2^2 \ominus b_1^2 \ominus b_2^2 \oplus 2a_1 a_2 i,\ 2a_1(b_1 \oplus b_2 i)\}.$$

Now, in order that $(\alpha, \beta)^2$ may be equal to $+1$, by the definition of our numbers, it is necessary and sufficient that the relations

$$\begin{cases} a_1^2 \ominus a_2^2 \ominus b_1^2 \ominus b_2^2 \oplus 2a_1 a_2 i = +1, \\ 2a_1(b_1 \oplus b_2 i)=0 \end{cases}$$

or the relations $\begin{cases} a_1^2 \ominus a_2^2 \ominus b_1^2 \ominus b_2^2 = +1 \text{ and } a_1 a_2=0, \\ a_1=0 \text{ or } b_1 \oplus b_2 i=0 \end{cases}$

may hold good. But here a_1 connot be zero, since, if so, we would have the relation $-(a_2^2 \oplus b_1^2 \oplus b_2^2)=+1$, which is impossible, a_2, b_1, b_2 being real. Therefore, we must have the relations $b_1 \oplus b_2 i=0$ and $a_2=0$, and accordingly must have the relations $b_1=0$, $b_2=0$, $a_2=0$, whence follows the relations $a_1^2=+1$ and $a_1=\pm 1$ at once. Thus the required numbers are $(\pm 1;\ 0)$ and only these.

Next, in order that $(\alpha, \beta)^2$ may be equal to -1, it is necessary and sufficient that the relations

$$\begin{cases} a_1^2 \ominus a_2^2 \ominus b_1^2 \ominus b_2^2 = -1 \text{ and } a_1 a_2=0, \\ a_1=0 \text{ or } b_1 \oplus b_2 i=0 \end{cases}$$

hold good. But, in this case, a_1 must be zero as may easily be seen. Therefore, we have the relation

$$a_2^2 \oplus b_1^2 \oplus b_2^2=+1.$$

Thus, in this case, a_2, b_1, b_2 may be considered to represent the direction cosines of a straight line in space. If a_2, b_1, b_2 are to be integers, only possible solutions of this equality are $a_2=\pm 1, b_1=0, b_2=0$; $b_1=\pm 1, a_2=0, b_2=0$, $b_2=\pm 1, a_2=0, b_1=0$. Therefore, the required numbers are $(\pm i,\ 0)$, $(0, \pm 1)$, $(0, \pm i)$, or $\pm i, \pm j, \pm k$. Thus it will be seen that, *in our mode of treatment, the four units of quaternions are represented in the most natural and simplest forms* $(+1,\ 0)$, $(+i,\ 0)$, $(0, +1)$, $(0, +i)$.

Relations of Four Units.

(i) $\quad i \otimes j=(i,\ 0) \otimes (0,\ 1)=(0,\ 1)=k;$

$\therefore \quad i \otimes j=k.$

$$j \otimes i=(0,\ 1) \otimes (i,\ 0)=(0, -i)=(-1,\ 0) \otimes (0, i)$$
$$= -1 \otimes k = -k;$$

$\therefore \quad j \otimes i = -k;$

$\therefore \qquad i \otimes j = -j \otimes i = k.$

(ii) $\qquad i \otimes k = (i, 0) \otimes (0, i) = (0, i^2) = (0, -1) = (-1, 0) \otimes (0, 1)$
$= -1 \otimes j = -j,$

$\therefore \qquad k \otimes i = (0, i) \otimes (i, 0) = \{0, i(-i)\} = (0, -i^2) = (0, 1) = +j;$

$\therefore \qquad k \otimes i = -i \otimes k = j.$

(iii) $\qquad j \otimes k = (0, 1) \otimes (0, i) = \{(-1)(-i), 0\} = (i, 0) = i,$

$k \otimes j = (0, i) \otimes (0, 1) = (-i, 0) = -i;$

$\therefore \qquad j \otimes k = -k \otimes j = i.$

(iv) $\qquad 1 \otimes j = j \otimes 1 = j; \; 1 \otimes k = k \otimes 1 = k; \; 1 \otimes i = i \otimes 1 = i.$

(v). When a_0 denotes a real number, we have the following relations:

(a) $\qquad a_0 \otimes j = (a_0, 0) \otimes (0, 1) = (0, a_0),$

$j \otimes a_0 = (0, 1) \otimes (a_0, 0) = (0, a_0);$

$\therefore \qquad a_0 \otimes j = j \otimes a_0$ and $(0, a_0) = a_0 j.$

(b) $\qquad a_0 \otimes k = (a_0, 0) \otimes (0, i) = (0, a_0 i),$

$k \otimes a_0 = (0, i) \otimes (a_0, 0) = (0, a_0 i);$

$\therefore \qquad a_0 \otimes k = k \otimes a_0$ and $(0, a_0 i) = a_0 k.$

Therefore, in this system of numbers, the commutative law does not hold good generally as is well known.

Recapitulation of Units.

Here we recapitulate the units of several systems of numbers to show how they are enlarged step by step, and moreover, to show that they are expressed in most natural and symmetrical form by our mode of treatment, namely by the form of "pair of numbers."

Only unit in the system of natural numbers is our fundamental first number $(A.A) \equiv 1$*; and in the system of integers, two units (positive and negative units) are derived from this unit, namely* $(1, 0) = +1$ *and* $(0, 1) = -1$. *Further, in the system of complex numbers, two new more units (positive and negative imaginary units) are derived from the above units* $+1$ *and* -1, *namely* $(+1, 0) = +1$, $(-1, 0) = -1$, $(0, +1) = +i$, $(0, -1) = -i$*; and in the system of quaternions, four new more units (positive and negative quaternion-units) are derived from the above units* ± 1 *and* $\pm i$, *namely* $(\pm 1, 0) = \pm 1$, $(\pm i, 0) = \pm i$, $(0, \pm 1) = \pm j$, $(0, \pm i) = \pm k$. They may be arranged in the following table:

Natural numbers	$(A.\ A)=1.$
Integers	$(1, 0)=+1,\ (0, 1)=-1.$
Complex numbers	$(+1, 0)=+1,\ (-1, 0)=-1,\ (0, +1)=+i,\ (0, -1)=-i.$
Quaternions	$(\pm 1, 0)=\pm 1,\ (\pm i, 0)=\pm i,\ (0, \pm 1)=\pm j,\ (0, \pm i)=\pm k.$

Fundamental theorem 1. Any quaternion may be expressed by a linear homogeneous function of the four units $(1, i, j, k)$ with real coefficients.

Lemma 1. When b_1 and b_2 denote any two real numbers, the relation

$$(b_1 \oplus b_2 i) \otimes j = b_1 j \oplus b_2 i j$$

always holds good.

Proof.
$$b_1 j \oplus b_2 ij = \{(b_1, 0) \otimes (0, 1)\} \oplus \{(b_2 i, 0) \otimes (0, 1)\}$$
$$=(0, b_1) \oplus (0, b_2 i) = (0, b_1 \oplus b_2 i),$$

and
$$(b_1 \oplus b_2 i) \otimes j = (b_1 \oplus b_2 i, 0) \otimes (0, 1) = (0, b_1 \oplus b_2 i);$$
$$\therefore \quad (b_1 \oplus b_2 i) = b_1 j \otimes b_2 i j.$$

Lemma 2. $(b_2 i) j = b_2 (i j).$

$$(b_2 i) j = (b_2 i, 0) \otimes (0, 1) = (0, b_2 i),$$
$$b_2 (i j) = (b_2, 0) \otimes \{(i, 0) \otimes (0, 1)\} = (b_2, 0) \otimes (0, i)$$
$$=(0, b_2 i),$$
$$\therefore \quad (b_2 i) j = b_2 (i j).$$

Lemma 3. $\{(\alpha, \beta) \oplus (\alpha', \beta')\} \oplus (\alpha'', \beta'') = (\alpha, \beta) \oplus \{(\alpha', \beta') \oplus (\alpha'', \beta'')\}.$

This follows at once from Postulate (I) and the property of complex numbers.

Proof of the theorem. Take any quaternion (α, β) and put $\alpha = a_1 \oplus a_2 i$, $\beta = b_1 \oplus b_2 i$, then we have

$$(\alpha, \beta) = (a_1 \oplus a_2 i,\ b_1 \oplus b_2 i)$$
$$=(a_1 \oplus a_2 i) \oplus (b_1 \oplus b_2 i) j \quad \text{(by Formula (3))}$$
$$=a_1 \oplus a_2 i \oplus b_1 j \oplus b_2 i j \quad \text{(by Lemmas (1) and (3))}$$

$$=a_1 \oplus a_2 i \oplus b_1 j \oplus b_2(ij) \qquad \text{(by Lemma (2))}$$
$$=a_1 \oplus a_2 i \oplus b_1 j \oplus b_2 k \qquad \text{(Formula (i))}.$$
$$\therefore \qquad (\alpha, \beta)=a_1 . 1 \oplus a_2 i \oplus b_1 j \oplus b_2 k.$$

Fundamental theorem 2. The sum and product of any two quaternions (α, β) *and* (α', β') *of our system can be obtained by adding and multiplying the corresponding linear expressions* $(a_1 \oplus a_2 i \oplus b_1 j \oplus b_2 k)$ *and* $(a_1' \oplus a_2' i \oplus b_1' j \oplus b_2' k)$ *by the ordinary methods of addition and multiplication in algebra.*

Proof. When (α, β) and (α', β') are expressed in the forms $(a_1 \oplus a_2 i, b_1 \oplus b_2 i)$ and $(a_1' \oplus a_2' i, b_1' \oplus b_2' i)$, they can also be expressed in the forms $(a_1 \oplus a_2 i \oplus b_1 j \oplus b_2 k)$ and $(a_1' \oplus a_2' i \oplus b_1' j \oplus b_2' k)$ by the fundamental theorem 1. Now first let us consider the sum of them. By Postulate (1), we have the relation

$$(\alpha, \beta) \oplus (\alpha', \beta') \equiv (\alpha \oplus \alpha', \beta \oplus \beta').$$

In this relation, substitute $a_1 \oplus a_2 i, b_1 \oplus b_2 i, a_1' \oplus a_2' i, b_1' \oplus b_2' i$ in places of $\alpha, \beta, \alpha', \beta'$ respectively and apply the fundamental theorem 1, then we have

$$(a_1 \oplus a_2 i, b_1 \oplus b_2 i) \oplus (a_1' \oplus a_2' i, b_1' \oplus b_2' i) \equiv \{(a_1 \oplus a_1') \oplus (a_2 \oplus a_2') i,$$
$$(b_1 \oplus b_1') \oplus (b_2 \oplus b_2') i\},$$

or
$$(a_1 \oplus a_2 i \oplus b_1 j \oplus b_2 k) \oplus (a_1' \oplus a_2' i \oplus b_1' j \oplus b_2' k) \equiv (a_1 \oplus a_1')$$
$$\oplus (a_2 \oplus a_2') i (b_1 \oplus b_1') j \oplus (b_2 \oplus b_2') k.$$

But the right-hand side of this expression is the same as that expression obtained by adding the two expressions of the left-hand side according to the law of the ordinary addition in algebra.

Next let us consider the product of two quaternions. By Postulate (II), we have the relation

$$(\alpha, \beta) \otimes (\alpha', \beta') \equiv (\alpha\alpha' \ominus \beta\bar{\beta}', \alpha\beta' \oplus \bar{\alpha}'\beta).$$

Substituting $\alpha = a_1 \oplus a_2 i$, $\beta = b_1 \oplus b_2 i$, $\alpha' = a_1' \oplus a_2' i$, $\beta' = b_1' \oplus b_2' i$ in this relation, and applying the fundamental theorem 1, we have the relation

$$(a_1 \oplus a_2 i \oplus b_1 j \oplus b_2 k) \otimes (a_1' \oplus a_2' i \oplus b_1' j \oplus b_2' k) \equiv$$
$$\{(a_1 a_1' \ominus a_2 a_2' \ominus b_1 b_1' \ominus b_2 b_2') \oplus (a_1 a_2' \oplus a_2 a_1' \oplus b_1 b_2' \ominus b_2 b_1') i \oplus$$
$$(a_1 b_1' \ominus a_2 b_2' \oplus b_1 a_1' \oplus b_2 a_2') j \oplus (a_1 b_2' \oplus a_2 b_1' \ominus b_1 a_2' \oplus b_2 a_1') k\}.$$

But the right-hand side of this expression is the same as that expression obtained by multiplying the two expressions of the left-hand side

according to the law of the ordinary multiplication in algebra, taking into account the fundamental relations existing between the units $1, i, j, k$.

From the relations of units and the above fundamental theorems, we know that our new system of numbers has all properties which characterise the ordinary quaternions; and so all properties of our numbers are deduced by the same method as that used in the ordinary quaternion.

Conclusion.

By the method of "pairs of numbers," we have enlarged the number-system, step by step, from the natural numbers up to the quaternions. By every step, a higher system is always obtained from the former system, including that former system as its subclass. Here we recapitulate these steps in tabular form to give a simple and clear view of them.

One fundamental thing A.			
Two fundamental numbers		$(A, A) \equiv A$,	(the former number)
		$(A, A) \equiv B$.	(new number)
Natural numbers	$\{(A, M)\}$	$(A, M)_{M \equiv A}$, $(A. M)_{M \equiv A}$	(the former fundamental numbers)
		$(A, M)_{M \not\equiv A}$	(new natural numbers)
Integers	$\{(a, b)\}$	$(a, b)_{a>b}$	(natural numbers)
		$(a, b)_{a \ngtr b}$ … $(a, b)_{a=b}$	(zero)
		$(a, b)_{a \ngtr b}$ … $(a, b)_{a<b}$	(negative integers)
Rational numbers	$\{(\alpha, \beta)\}$	$(\alpha, \beta)\ \alpha = \beta \otimes \gamma$	(integers)
		$(\alpha, \beta)\ \alpha \neq \beta \otimes \gamma$	(fractions)
Real numbers	$\{(\mathfrak{A}, \mathfrak{B})\}$	$(\mathfrak{A}, \mathfrak{B})_{\text{having a characteristic element}}$	(rational numbers)
		$(\mathfrak{A}, \mathfrak{B})_{\text{having no characteristic element}}$	(irrational numbers)
Complex numbers	$\{(a, b)\}$	$(a, b)_{b=0}$	(real numbers)
		$(a, b)_{b \neq 0}$ … $(a, b)_{a=0}$.	(imaginary numbers)
		$(a, b)_{b \neq 0}$ … $(a, b)_{a \neq 0}$	(essential complex numbers)
Quaternions	$\{(\alpha, \beta)\}$	$(\alpha, \beta)_{\beta=0}$	(complex numbers)
		$(\alpha, \beta)_{\beta \neq 0}$	(ordinary quaternions)

Note sur quelques identités vectorielles et algébriques; et sur les formules de la trigonométrie sphérique,

par

Georges Tiercy, Genève, Suisse.

On ne dira jamais assez combien la résolution des problèmes de géométrie sphérique devient simple, lorsqu'on utilise les notations vectorielles. Une identité vectorielle, en effet, des qu'interprétée, conduit à une formule de trigonométrie sphérique, le nombre des vecteurs figurant dans la relation donnant le nombre des sommets du polygone ; il en résulte qu' une seule identité conduit à une série de formules. D'autre part, ces identités donnent des démonstrations rapides de théorèmes de géométrie sphérique([1]) ; et enfin, les relations scalaires fournissent des identités algébriques.

1. On connaît la définition du *produit extérieur* ou *vectoriel* de deux vecteurs r et r_1, dont les projections sur trois axes rectangulaires sont respectivement $(x\,;\ y\,;\ z)$ et $(x_1\,;\ y_1\,;\ z_1)$; c'est un troisième vecteur e, perpendiculaire au plan des deux premiers, et dont la longueur est donnée par la relation

$$e=[rr_1]=rr_1 \sin a,$$

a étant l'angle de r et r_1 ; ses projections sur les axes sont :

$$e_x=yz_1-y_1z\,; \qquad e_y=zx_1-z_1x\,; \qquad e_z=xy_1-x_1y.$$

D'autre part, le *produit intérieur* ou *scalaire* de r et r_1 est un nombre donné par l'expression :

$$(rr_1)=xx_1+yy_1+zz_1\,;$$

il est égal à $rr_1 \cos a$.

Pour ménager la place, nous avons supprimé toute figure ; le lecteur fera bien de s'aider d'un croquis.

Nous utiliserons l'abréviation :

$$(rr_1r_2)=(r[r_1r_2])\,;$$

et nous designerons par $\{h^A_{BC}\}$ l'arc de grand cercle mené, sur la sphère, d'un point A au grand cercle BC, perpendiculairement à BC.

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(1) Lire, sur ce sujet, l'article que M. Fr. Daniels a publié dans *l'Enseignement mathématique*, tome 20, n°-2.

2. Rappelons tout d'abord que les identités vectorielles bien connues

$$(\alpha\beta\gamma) = (\gamma\alpha\beta) = (\beta\gamma\alpha) \tag{1}$$

donnent immédiatement, en simplifiant par le facteur $|\alpha\beta\gamma|$, et en appelant $(A; B; C)$ les traces des vecteurs sur la sphère de rayon unité, un groupe de formules de trigonométrie sphérique élémentaire :

$$\sin a.\, \sin h_{BC}^{A} = \sin b.\, \sin h_{AC}^{B} = \sin c.\, \sin h_{AB}^{C}\,; \tag{1'}$$

les longueurs des vecteurs ne jouent aucun rôle.

On sait d'ailleurs que les trois hauteurs sphériques $h_{BC}^{A}, h_{AC}^{B}, h_{AB}^{C}$ sont concourantes ; en effet, on a l'identité

$$[[\alpha\beta]\gamma] = (\alpha\gamma)\beta - (\beta\gamma)\alpha, \tag{2}$$

dans laquelle chaque membre exprime un vecteur parallèle au plan de α et β, et perpendiculaire à γ ; si l'on permute circulairement les lettres α, β, γ, et si l'on fait la somme des trois identités que l'on obtient ainsi, on trouve :

$$[[\beta\gamma]\alpha] + [[\gamma\alpha]\beta] + [[\alpha\beta]\gamma] = 0.$$

3. Multiplions l'identité (2) scalairement par un nouveau vecteur δ, en tenant compte de (1) ; on obtient la relation

$$([\alpha\beta][\gamma\delta]) = (\alpha\gamma)(\beta\delta) - (\beta\gamma)(\alpha\delta). \tag{3}$$

Elle conduit, pour les quadrilatères sphériques, à la formule très simple que voici

$$\sin AB.\, \sin CD.\, \cos i = \cos AC.\, \cos BD - \cos AD.\, \cos BC, \tag{3'}$$

ou i est l'angle des plans AOB et COD.

Si le vecteur β se confond avec δ, on retrouve la formule fondamentale des triangles :

$$\left\{\begin{array}{c} \sin AB.\, \sin BC.\, \cos i = \cos AC - \cos AB.\, \cos BC \\ \text{ou :} \\ \cos b = \cos a.\, \cos c + \sin a.\, \sin c.\, \cos B. \end{array}\right. \tag{3''}$$

Enfin, il est facile de voir que l'identité (3) et la formule trigonométrique (3') tout les productions géométriques de l'identité algébrique suivante :

$$\left\{\begin{aligned} &\begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} . \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} + \begin{vmatrix} a'' & a \\ b'' & b \end{vmatrix} . \begin{vmatrix} c'' & c \\ d'' & d \end{vmatrix} + \begin{vmatrix} a & a' \\ b & b' \end{vmatrix} . \begin{vmatrix} c & c' \\ d & d' \end{vmatrix} = \\ &\qquad = \begin{vmatrix} ac + a'c' + a''c'' & ad + a'd' + a''d'' \\ bc + b'c' + b''c'' & bd + b'd' + b''d \end{vmatrix}; \end{aligned}\right.$$

il suffit de considérer quatre vecteurs $\alpha, \beta, \gamma, \delta$, dont les composantes sont respectivement $(a\,;\, a';\, a'')$, $(b\,;\, b';\, b'')$, $(c\,;\, c';\, c'')$ et $(d,\, d',\, d'')$.

Remarquons en outre que l'identité (3) donne :

$$([\alpha\beta][\alpha\beta])=\alpha^2\beta^2-(\alpha\beta)^2, \qquad \text{ou :} \qquad \sin^2 AOB+\cos^2 AOB=1.$$

4. Prenons le cas où, dans l'identité (2), les vecteurs β et γ sont perpendiculaires entre eux ; on a alors $(\beta\gamma)=0$; et la relation (2) devient :

$$[[\alpha\beta]\,\gamma]=(\alpha\gamma)\beta\,;$$

multiplions scalairement par β ; il vient :

$$([\alpha\beta][\gamma\beta])=(\alpha\gamma)(\beta\beta)\,; \tag{4}$$

et par suite, pour le triangle sphérique ABC :

$$\sin AB.\cos B=\cos AC.$$

Menons BD perpendiculaire sur CA ; et appelons $\mathfrak{B}$ l'angle ABD ; on obtient pour le triangle rectangle ABD :

$$\sin AB.\sin \mathfrak{B}=\sin AD. \tag{4'}$$

Pour un triangle quelconque MNP, on pourra donc écrire :

$$\sin m.\sin N=\sin n.\sin M, \tag{4''}$$

formule élémentaire bien connue. Il est d'ailleurs à remarquer que les relations (4″) sont contenues dans les relations (3″).

5. Reprenons l'identité (3), avec des vecteurs μ, ν, ρ, σ :

$$([\mu\nu][\rho\sigma])=(\mu\rho)(\nu\sigma)-(\mu\sigma)(\nu\rho)\,;$$

et supposons que ces vecteurs aient les valeurs suivantes :

$$\mu=[\alpha\beta,]\;\; \nu=[\gamma\delta],\;\; \rho=[\alpha\delta],\;\; \sigma=[\beta\gamma],$$

$\alpha, \beta, \gamma, \delta$ étant quatre nouveaux vecteurs. On obtient une nouvelle identité :

$$\begin{aligned}([[\alpha\beta][\gamma\delta]]\,[[\alpha\delta][\beta\gamma]])=&([\alpha\beta][\alpha\delta])\,([\gamma\delta][\beta\gamma])\\ &-([\alpha\beta][\beta\gamma])\,([\gamma\delta][\alpha\delta]).\end{aligned} \tag{5}$$

Cette identité vectorielle correspond à l'identité algébrique suivante :

$$\sum_{\text{perm.}}\begin{vmatrix} a''b-ab'' & ab'-a'b \\ c''d-cd'' & cd'-c'd\end{vmatrix}.\begin{vmatrix} a''d-ad'' & ad'-a'd \\ b''c-bc'' & bc'-b'c\end{vmatrix}=$$

$$=\begin{vmatrix} \sum\limits_{\text{perm.}}(a'b''-a''b')(a'd''-a''d') & \sum\limits_{\text{perm.}}(a'b''-a''b')(b'c''-b''c') \\ \sum\limits_{\text{perm.}}(c'd''-c''d')(a'd''-a''d') & \sum\limits_{\text{perm.}}(c'd''-c''d')(b'c''-b''c')\end{vmatrix}.$$

En désignant par I l'intersection de AD et BC sur la sphère, par J celle de AB et CD, et par d l'arc IJ, l'identité (5) conduit à la relation :

$$\sin I . \sin J . \cos d = \cos A . \cos C - \cos B . \cos D, \qquad (5')$$

formule importante relative au quadrilatère $ABCD$.

Dans le cas particulier où les vecteurs $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ sont coplanaires, l'angle B vaut π, et on obtient :

$$\cos D = -\cos A . \cos C + \sin A . \sin C . \cos d, \qquad (5'')$$

autre formule fondamentale des triangles sphériques.

6. Prenons encore l'identité (3) ; et faisons-y :

$$\gamma = [\alpha \varepsilon] \text{ et } \delta = [\varepsilon \varphi] ;$$

comme $(\alpha [\alpha \varepsilon]) = 0$, il vient :

$$([\alpha \beta] . [[\alpha \varepsilon][\varepsilon \varphi]]) = -(\alpha [\varepsilon \varphi])(\beta [\alpha \varepsilon]). \qquad (6)$$

La relation (3) donne :

$$\sin AOB . \sin COD . \cos i = \cos AOC . \cos BOD - \cos AOD . \cos BOC,$$

où i est l'angle des plans AOB et COD. Passant à l'identité (6), on trouve :

$$\begin{cases} \sin COD = \sin AEF ; & \cos AOD = \sin h^A_{EF} \quad ; \\ \quad AOC = \pi/2 \quad ; & \cos BOC = -\sin h^B_{AE} ; \end{cases}$$

$$\sin AB . \sin AEF . \cos i = \sin h^A_{EF} . \sin h^B_{AE} ;$$

d'ailleurs, les formules (4'') donnent ici :

$$\begin{cases} \sin h^A_{EF} = \sin AF . \sin F = \sin AE . \sin AEF \quad ; \\ \sin h^B_{AE} = \sin AB . \sin BAE = \sin BE . \sin AEB ; \end{cases}$$

puis, en appelant P la trace, sur la sphère, du troisième vecteur de α et β ; et en tenant compte du fait que E est la trace du 3e-vecteur de γ et δ, on a :

$$\cos i = \cos POE = \sin h^E_{AB} = \sin BE . \sin B ;$$

on obtient donc finalement :

$$\begin{cases} \sin BE . \dfrac{\sin B}{\sin BAE} \sin = AF . \dfrac{\sin F}{\sin AEF} ; \\ \sin AB . \dfrac{\sin B}{\sin AEB} = \sin AF . \dfrac{\sin F}{\sin AEF} . \end{cases} \qquad (6)'$$

Considérons le cas particulier où le vecteur φ tend vers la position de ε, en se déplaçant dans le plan FOE ; AF devient AE ; $\sin F$ tend vers $\sin E$, de telle sorte qu'on a :

$$\lim \frac{\sin F}{\sin E} = 1 ;$$

et nous retombons sur les formules de triangles :

$$\begin{cases} \sin BE . \sin B = \sin AE . \sin A ; \\ \sin AB . \sin B = \sin AE . \sin E ; \end{cases}$$

il en serait d'ailleurs de même si le vecteur φ était simplement coplanaire avec α et ε, sans coïncider avec ε.

Enfin, l'identité (6) et les formules trigonométriques (6′) sont la traduction géometrique de l'identité algébrique suivante :

$$\sum_{\text{perm.}} (a'b''-a''b') . \begin{vmatrix} a''e-ae'' & ae'-a'e \\ e''f-ef'' & ef'-e'f \end{vmatrix} = \begin{vmatrix} a & a' & a'' \\ e & e' & e'' \\ f & f' & f'' \end{vmatrix} . \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ e & e' & e'' \end{vmatrix} .$$

7. Partons encore de l'identité (3) :

$$(\alpha\gamma)(\beta\delta) - (\alpha\beta)(\gamma\delta) = ([\alpha\delta][\beta\gamma]) ;$$

et faisons $\delta = \varphi - \varepsilon$, où l'on pourra supposer que les vecteurs ε et φ ont la même longueur ; cela donne sur la sphère un pentagone $ABCEF$; il vient :

$$(\alpha\gamma)(\beta\{\varphi-\varepsilon\}) - (\alpha\beta)(\gamma\{\varphi-\varepsilon\}) = ([\alpha\{\varphi-\varepsilon\}][\beta\gamma]).$$

Or, le troisième vecteur $(\alpha\delta)$ est la résultante des deux troisièmes vecteurs $(\alpha\varphi)$ et $(-\alpha\varepsilon)$; en effet, la projection de l'aire AOD sur un plan quelconque est égale à la différence des projections des aires AOF et AOE. On a donc la différence géométrique :

$$\alpha\delta \sin AOD = \alpha\varphi \sin AOF - \alpha\varepsilon \sin AOE ;$$

comme, d'autre part, le produit intérieur jouit de la propriété distributive, on peut écrire, après simplification par $\alpha\beta\gamma$, et en remarquant qu'on peut toujours se placer dans le cas où $|\varphi| = |\varepsilon|$:

$$\begin{cases} \cos AC (\cos BF - \cos BE) - \cos AB (\cos CF - \cos CE) \\ = \sin BC (\sin AF . \cos i' - \sin AE . \cos i''), \end{cases} \qquad (7')$$

où i' et i'' sont les angles que forme le plan BOC respectivement avec les plans AOF et AOE.

Si le pentagone $ABCEF$ est régulier, on aura, en appelant c le côté et d la diagonale :

$$\cos^2 c - \sin^2 c . \cos i' = \cos c . \cos d - \sin c . \sin d . \cos i'' \qquad (7'')$$

où i' est l'angle de deux côtés non consécutifs, et i'' l'angle d'un côté avec la diagonale opposée.

Le rayon de la sphère étant connu, si l'on donne l'un des quatre éléments (c, d, i', i''), les trois autres s'en déduiront ; la relation $(7'')$ servira à déterminer le quatrième.

8. Appliquons deux fois le théorème du double produit vectoriel (2) ; et partons des identités :

$$[[\alpha\beta][\gamma\delta]] = (\alpha\gamma\delta)\beta - (\beta\gamma\delta)\alpha = (\delta\alpha\beta)\gamma - (\gamma\alpha\beta)\delta. \qquad (8)$$

Multiplions scalairement par $\pi = [[\alpha\beta][\gamma\delta]]$; il vient :

$$\left\{\begin{aligned} [[\alpha\beta][\gamma\delta]]^2 &= (\alpha\gamma\delta)(\beta[\alpha\beta][\gamma\delta]) - (\beta\gamma\delta)(\alpha[\alpha\beta][\gamma\delta]) \\ &= (\delta\alpha\beta)(\gamma[\alpha\beta][\gamma\delta]) - (\gamma\alpha\beta)(\delta[\alpha\beta][\gamma\delta]). \end{aligned}\right. \qquad (9)$$

Appelant P l'intersection de AB et CD ; en tenant compte de $(4')$ et $(4'')$, la traduction trigonométrique donne, après simplification par le facteur $(\alpha^2\beta^2\gamma^2\delta^2 \sin AB . \sin CD . \sin P)$, les formules :

$$\left\{\begin{aligned} \sin AB . \sin P &= \sin AD . \cos BP . \sin D - \sin BC . \cos AP . \sin C ; \\ \sin CD . \sin P &= \sin AD . \cos CP . \sin A - \sin BC . \cos DP . \sin B . \end{aligned}\right. \qquad (9')$$

D'ailleurs, si l'on considère le cas particulier où le quadrilatère $ABCD$ se réduit à un triangle, les formules ci-dessus reconduisent à $(4'')$.

Les identités (9) et les formules trigonométriques qui en découlent correspondent aux identités algébriques suivantes :

$$\left\{\begin{aligned} &\sum_{\text{perm.}} \begin{vmatrix} a''b - ab'' & ab' - a'b \\ c''d - cd'' & cd' - c'd \end{vmatrix}^2 \\ &= \begin{vmatrix} \sum\limits_{\text{perm.}} a(c'd'' - c''d') & \sum\limits_{\text{perm.}} b(c'd'' - c''d') \\ \sum\limits_{\text{perm.}} a \begin{vmatrix} a''b - ab'' & ab' - a'b \\ c''d - cd'' & cd' - c'd \end{vmatrix} & \sum\limits_{\text{perm.}} b \begin{vmatrix} a''b - ab'' & ab' - a'b \\ c''d - cd'' & cd' - c'd \end{vmatrix} \end{vmatrix} \\ &= \begin{vmatrix} \sum\limits_{\text{perm.}} d(a'b'' - a''b') & \sum\limits_{\text{perm.}} c(a'b'' - a''b') \\ \sum\limits_{\text{perm.}} d \begin{vmatrix} a''b - ab'' & ab' - a'b \\ c''d - cd'' & cd' - c'd \end{vmatrix} & \sum\limits_{\text{perm.}} c \begin{vmatrix} a''b - ab'' & ab' - a'b \\ c''d - cd'' & cd' - c'd \end{vmatrix} \end{vmatrix}. \end{aligned}\right.$$

9. Les identités (8) donnent encore :

$$\begin{cases}[[\alpha\beta][\alpha\gamma]]=(\beta\gamma\alpha)\alpha=(\alpha\beta\gamma)\alpha=(\gamma\alpha\beta)\alpha\,; \\ [[\beta\gamma][\beta\alpha]]=(\alpha\beta\gamma)\beta\,;\ [[\gamma\alpha][\gamma\beta]]=(\alpha\beta\gamma)\gamma.\end{cases} \tag{10}$$

Multiplions scalairement ces trois identités, respectivement par $[\beta\gamma]$, $[\gamma\alpha]$ et $[\alpha\beta]$. On obtient :

$$\begin{aligned}([[\alpha\beta][\alpha\gamma]][\beta\gamma])&=([[\beta\gamma][\beta\alpha]]\,.\,[\gamma\alpha]) \\ &=([[\gamma\alpha][\gamma\beta]]\,.\,[\alpha\beta])\,; \end{aligned}\tag{11}$$

ce qui conduit à :

$$\begin{aligned}\sin AB.\sin AC.\sin A&=\sin BC.\sin BA.\sin B \\ &=\sin CA.\sin CB.\sin C,\end{aligned} \tag{11'}$$

formules fondamentales des triangles ; elles auraient pu être déduites de (1′) et (4′). Remarquons d'ailleurs qu'en multipliant scalairement les relations (10) respectivement par α, β, γ, on retrouve les formules (4′) et (4″).

10. Prenons encore les identitès (8), et multiplions-les scalairement par ε ; on a :

$$([[\alpha\beta][\gamma\delta]]\,.\,\varepsilon)=(\alpha\gamma\delta)(\beta\varepsilon)-(\beta\gamma\delta)(\alpha\varepsilon)=(\delta\alpha\beta)(\gamma\varepsilon)-(\gamma\alpha\beta)(\delta\varepsilon)\,; \tag{12}$$

appelant P et $\mathfrak{P}$ respectivement l'intersection et l'angle de AB et CD, cela donne pour le pentagone convexe $AEBCD$, après simplification par $\alpha\beta\gamma\delta\varepsilon$:

$$\left.\begin{aligned}\sin AB.\cos EP.\sin\mathfrak{P}&=\sin BC.\cos AE.\sin C \\ &\quad-\sin AD.\cos BE.\sin D\,; \\ \sin CD.\cos EP.\sin\mathfrak{P}&=\sin BC.\cos DE.\sin ABC \\ &\quad-\sin AD.\cos CE.\sin BAD\,; \\ \frac{\sin AB}{\begin{vmatrix}\sin AD.\sin D & \sin BC\sin C \\ \cos AE & \cos BE\end{vmatrix}} &= \frac{\sin CD}{\begin{vmatrix}\sin BC.\sin ABC & \sin AD.\sin BAD \\ \cos CE & \cos DE\end{vmatrix}}\end{aligned}\right\} \tag{12'}$$

on retrouvera des formules déjà connues en supposant que le pentagone se réduite à un quadrilatére.

Les identités (12) correspondent aux identités algébriques suivantes :

$$\sum_{\text{perm.}} e \begin{vmatrix} a''b-ab'' & ab'-a'b \\ c''d-cd'' & cd'-c'd \end{vmatrix} = \begin{vmatrix} \sum\limits_{\text{perm.}} a(c'd''-c''d') & \sum\limits_{\text{perm.}} b(c'd''-c''d') \\ \sum\limits_{\text{perm.}} ae & \sum\limits_{\text{perm.}} be \end{vmatrix}$$

$$= \begin{vmatrix} \sum\limits_{\text{perm.}} d(a'b''-a''b') & \sum\limits_{\text{perm.}} c(a\,b''-a\,'b') \\ \sum\limits_{\text{perm.}} de & \sum\limits_{\text{perm.}} ce \end{vmatrix}.$$

11. Toutes les identités utilisées jusqu'ici dérivent de l'identité fondamentale du double produit vectoriel. Prenons maintenant une autre identité entre quatre vecteurs :

$$(\beta\gamma\delta)\alpha - (\gamma\delta\alpha)\beta + (\delta\alpha\beta)\gamma - (\alpha\beta\gamma)\delta = 0\,; \tag{13}$$

elle exprime que trois quelconques des quatre vecteurs $\alpha, \beta, \gamma, \delta$, ne sont pas coplanaires ; elle peut s'écrire :

$$(\alpha\beta\gamma)\delta = (\delta\beta\gamma)\alpha + (\delta\gamma\alpha)\beta + (\delta\alpha\beta)\gamma.$$

Multiplions-la scalairement par ε ; il vient :

$$(\alpha\beta\gamma)(\delta\epsilon) = (\delta\beta\gamma)(\alpha\varepsilon) + (\delta\gamma\alpha)(\beta\varepsilon) + (\delta\alpha\beta)(\gamma\varepsilon),$$

dont la traduction trigonométrique est immédiate ; soit le pentagone convexe $ABCDE$; on a :

$$\sin BC \sin h^A_{BC}.\cos DE = \sin BC.\sin h^D_{BC}.\cos AE + \sin CA.\sin h^D_{CA}.\cos BE$$
$$+ \sin AB.\sin h^D_{AB}.\cos CE,$$

ou bien :

$$\sin AB.\sin BC.\sin B.\cos DE = \sin BC.\sin CD.\sin C.\cos AE$$
$$+ \sin CA.\sin CD.\sin ACD.\cos BE$$
$$+ \sin AB.\sin BD.\sin ABD.\cos CE.$$

L'identité algébrique correspondante est la suivante :

$$\left[\sum_{\text{perm.}} a(b'c''-b''c')\right]\left[\sum_{\text{perm.}} de\right] = \left[\sum_{\text{perm.}} d(b'c''-b''c')\right]\left[\sum_{\text{perm.}} ae\right]$$

$$+ \left[\sum_{\text{perm.}} d(c'a''-a'c'')\right]\left[\sum_{\text{perm.}} bc\right] + \left[\sum_{\text{perm.}} d(a'b''-a''b')\right]\left[\sum_{\text{perm.}} ce\right].$$

Supposons enfin que ε coïncide avec δ ; le pentagone se réduit au quadrilatère $ABCD$; et l'on a :

$$\sin AB.\sin BC.\sin B=\sin BC.\sin CD.\sin C.\cos AD+\sin CA.\sin CD.$$
$$\times\sin ACD.\cos AB+\sin AB.\sin BD.\sin ABD.\cos AC. \quad (14'')$$

12. Après multiplication scalaire par $[\varepsilon\varphi]$, l'identité (13) donne encore :

$$\left\{\begin{array}{l}(\beta\gamma\delta)(\alpha\varepsilon\varphi)-(\gamma\delta\alpha)(\beta\varepsilon\varphi)+(\delta\alpha\beta)(\gamma\varepsilon\varphi)-(\alpha\beta\gamma)(\delta\varepsilon\varphi)=0,\\ \text{ou :}\\ (\alpha\beta\gamma)(\delta\varepsilon\varphi)=(\delta\beta\gamma)(\alpha\varepsilon\varphi)+(\delta\gamma\alpha)(\beta\varepsilon\varphi)+(\delta\alpha\beta)(\gamma\varepsilon\varphi).\end{array}\right. \quad (15)$$

Après division par le facteur commun $(\alpha\beta\gamma\varepsilon\varphi.\sin EF)$, on trouvera pour l'hexagone convexe $ABCDEF$ les formules :

$$\left\{\begin{array}{l}\sin AB.\sin BC.\sin B.\sin DE\sin E=\sin BC.\sin CD.\sin C.\times\\ \sin AF.\sin F+\sin CA.\sin CD.\sin ACD.\sin BF.\sin BFE\\ +\sin AB.\sin BD.\sin ABD.\sin CE.\sin CEF;\end{array}\right. \quad (15')$$

et :

$$\left\{\begin{array}{l}\sin BC.\sin CD.\sin C.\sin AF.\sin F-\sin DA.\sin CD.\sin CDA.\times\\ \sin BF.\sin BFE+\sin AB.\sin BD.\sin ABD.\sin CE.\sin CEF\\ -\sin AB.\sin BC.\sin B.\sin DE.\sin E=0.\end{array}\right. \quad (15'')$$

Si l'on prend le cas particulier où le vecteur φ coïncide avec α, par exemple, il vient pour le pentagone $ABCDE$:

$$\sin BC.\sin B.\sin DE.\sin E=-\sin CA.\sin CD.\sin ACD.\sin A$$
$$+\sin BD.\sin ABD.\sin CE\sin CEA,$$

et :

$$\sin BD.\sin ABD.\sin CE.\sin CEA-\sin DA.\sin CD.\sin CDA.\sin A$$
$$=\sin BC.\sin B.\sin DE.\sin E;$$

comparant ces deux dernières relations, on retrouve la formule suivante pour le triangle ACD :

$$\sin CA.\sin CD.\sin ACD=\sin DA.\sin DC.\sin ADC.$$

Enfin, la traduction algébrique de l'identité (15) et des formules trigonométriques qui en découlent est l'identité suivante :

$$\begin{vmatrix}\sum\limits_{\text{perm.}} b(c'd''-c''d') & \sum\limits_{\text{perm.}} c(d'a''-d''a')\\ \sum\limits_{\text{perm.}} b(e'f''-e''f') & \sum\limits_{\text{perm.}} a(e'f''-e''f')\end{vmatrix}$$
$$+\begin{vmatrix}\sum\limits_{\text{perm.}} d(a'b''-a''b') & \sum\limits_{\text{perm.}} a(b'c''-b''c')\\ \sum\limits_{\text{perm.}} d(e'f''-e''f') & \sum\limits_{\text{perm.}} c(e'f''-e''f')\end{vmatrix}=0.$$

13. Les quelques exemples traités, tout élémentaires qu'ils soient montrent d'une façon bien nette que toute la trigonométrie sphérique ordinaire est contenue dans quelques identités vectorielles. Au total, nous n'avons utilisé que trois identités vectorielles fondamentales (1 ; 2 ; et 13) ; il a suffi de les interpréter de différentes façons pour en tirer une série de théorèmes.

Le lecteur pourra multiplier les exemples, en partant d'autres identités vectorielles, dont chacune conduira à une famille de formules trigonométriques et à un groupe d'identités algébriques ; en voici quelques-unes :

$$\begin{cases} \delta(\alpha\beta\gamma) = [\alpha\beta](\gamma\delta) + [\beta\gamma](\alpha\delta) + [\gamma\alpha](\beta\delta) ; \\ ([\alpha\beta][\beta\gamma][\gamma\alpha]) = -(\alpha\beta\gamma)^2 ; \\ (\alpha\beta\gamma) \cdot (\alpha_1\beta_1\gamma_1) = -\| \alpha\alpha_1 \| ; \\ \text{etc.} \end{cases}$$

Genève, janvier 1922.

On Liquid Motion inside a Rotating Elliptic Quadrant,

by

NRIPENDRANATH SEN, Calcutta, India.

The problem of the two dimensional motion of a homogeneous incompressible fluid inside rotating cylinders has attracted considerable attention from many eminent mathematicians. Professors Greenhill[1], Basset[2], Ferrers[3], Hicks[4], Filon[5] and others have completely solved several such problems. The object of the present paper[6] is to present a solution of the liquid motion inside a rotating elliptic quadrant. Towards the end of the paper I have shown that the liquid motion inside a rotating circular quadrant, discussed by Sir A. G. Greenhill in a paper already cited, can be easily deduced from the results obtained by me. Also a few alternative solutions have been obtained by the aid of some identities proved by me in a recent issue of the Bulletin[7] of the Calcutta Mathematical Society.

It may be noted here that the hydrodynamical solutions obtained in the paper also furnish solutions of the torsion problems in the Theory of Elasticity for the same boundaries.

1. For convenience, the liquid is supposed to be contained in a cylinder of unit length and confined between two smooth parallel planes at right angles to the axis of the cylinder. The cylinder is supposed to be rotating about the axis passing through the centre of the elliptic section. The corresponding result for rotation about any other point can be written down very easily (See Quart. Jour. Vol 17, 234, 1881).

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(1) Greenhill, "Rotating quadrantal sector of a circle", Mess. Math. Vol. 8, 89, 1879; also Mess. Math. Vol. 9, 35, 1880; also Quart. Jour. Vol. 16, 227, 1880, etc.

(2) Basset, Quart. Journ. 20, 234. 1884; also Do 19, 190, 1883; also 21, 336, 1885 etc.

(3) Ferrers, "Certain cylindrical vessels", Quart. Jour. Vol. 17, (227–44), 1881.

(4) Hicks, "Rotating semicircle", Mess. Math. Vol. 8, 42, 1879.

(5) Filon, "Resistance to torsion of certain forms of shafting", Phil-Trans, 193 A, 309, 1900. Also Gronwall, "On the influence of key ways etc." Trans Americ. Math. Soc. 20, 213, 1919.

(6) Some of the results of this paper were obtained by me as early as 1919 and read before the Calcutta Mathematical Society on the 27th July 1919.

(7) "Liquid motion inside certain rotating curvilinear rectangles"—Bull. Cal. Math. Soc. Vol. 9, 7, 1920.

2. Let $x+iy=c\cosh(\xi+i\eta)$, $\omega=$ angular velocity,
$\xi=\alpha$ on the ellipse, $\psi=$ stream function,

then $\psi=\frac{\omega c^2}{4}(\cosh 2\xi+\cos 2\eta)$ on the boundaries of the elliptic quadrant

$$\begin{aligned} &(A)\quad \xi=\alpha, \quad 0\leqq\eta\leqq\frac{\pi}{2},\\ &(B)\quad \eta=\frac{\pi}{2}, \quad 0\leqq\xi\leqq\alpha,\\ &(C)\quad \xi=0, \quad 0\leqq\eta\leqq\frac{\pi}{2},\\ &(D)\quad \eta=0, \quad 0\leqq\xi\leqq\alpha. \end{aligned} \tag{1}$$

Assume

$$\psi=\frac{\omega c^2}{4}\left[-\frac{4}{\pi}\left\{\left(\eta-\frac{\pi}{4}\right)\cosh 2\xi\cos 2\eta+(\xi-\alpha)\sinh 2\xi\sin 2\eta\right.\right.$$
$$\left.\left.+\left(\eta-\frac{\pi}{4}\right)\right\}+\sum_{m=1}^{\infty}\left\{A_m\sinh 2m(\xi-\alpha)+B_m\sinh 2m\xi\right\}\sin 2m\eta\right]. \quad (2)$$

Evidently $\nabla^2\psi=0$ and the boundary conditions (B) and (D) are satisfied identically. Now, to determine A_m, B_m, so as to satisfy the boundary conditions (A) and (C), putting $\xi=0$ in (1) and (2) and equating those, we have after a little simplification,

$$\Sigma A_m\sinh 2m\alpha\sin 2m\eta=-\frac{4\eta}{\pi}(1+\cos 2\eta),$$

where
$$0\leqq\eta\leqq\frac{\pi}{2}.$$

Now multiplying both sides by $\sin 2m\eta\, d\eta$ and integrating between the limits 0 and $\frac{\pi}{2}$, we have

$$\frac{\pi}{4}A_m\sinh 2m\alpha=(-1)^{m+1}\frac{1}{2}\left[\frac{1}{m+1}+\frac{1}{m-1}-\frac{2}{m}\right].$$

Therefore

$$A_m=(-1)^{m+1}\frac{2}{\pi\sinh 2m\alpha}\left[\frac{1}{m+1}+\frac{1}{m-1}-\frac{2}{m}\right]. \quad (3)$$

Putting $\xi=\alpha$ in (1) and (2) and proceeding exactly as above,

$$B_m=\frac{1}{\pi\sinh 2m\alpha}\left[\{(-1)^m+1\}-\{(-1)^{m+1}+1\}\cosh 2\alpha\right]$$

$$\cdot\left(\frac{1}{m+1}+\frac{1}{m-1}-\frac{2}{m}\right). \qquad (4)$$

Hence

$$B_{2m}=\frac{2}{\pi\sinh 4m\alpha}\left[\frac{1}{2m+1}+\frac{1}{2m-1}-\frac{1}{m}\right]$$

and

$$B_{2m+1}=\frac{2\cosh 2\alpha}{\pi\sinh 2(2m+1)\alpha}\left[\frac{1}{2(m+1)}+\frac{1}{2m}-\frac{2}{2m+1}\right].$$

Now, (3) and (4) give A_m, B_m except when $m=1$. But A_1 and B_1 can be obtained by putting $\xi=0$ and α in (1) and (2) respectively, equating and multiplying by $\sin 2\eta\, d\eta$ and then integrating between the limits 0 and $\frac{\pi}{2}$. This will give

$$A_1=-\frac{3}{\pi\sinh 2\alpha},\quad B_1=\frac{3}{\pi}\cosh 2\alpha. \qquad (5)$$

Thus ψ is completely determined from (2), (3), (4) and (5).

3. Remembering that $\phi+i\psi=f(\xi+i\eta)$, the expression for the velocity potential ϕ can be obtained from (2) by inspection:

$$\phi=\frac{\omega c^2}{4}\left[-\frac{4}{\pi}\left\{(\xi-\alpha)\cosh 2\xi\cos 2\eta-\left(\eta-\frac{\pi}{4}\right)\sinh 2\xi\sin 2\eta+\xi\right\}\right.$$

$$\left.+\sum_{m=1}^{\infty}\{A_m\cosh 2m(\xi-\alpha)+B_m\cosh 2m\xi\}\cos 2m\eta\right]. \qquad (6)$$

4. Let T be the kinetic energy, then

$$T=-\frac{\rho}{2}\iint\phi\frac{\partial\phi}{\partial n}ds \text{ (taken over the boundaries } (A), (B), (C) \text{ and } (D))$$

$$=\frac{\rho}{2}\int\phi\, d\psi$$

$$=\frac{\rho}{2}\left[\int_{\xi=0}^{\xi=\frac{\pi}{2}} \phi_a\, d\psi_a+\int_{\xi=a}^{0} \phi_{\frac{\pi}{2}}\, d\psi_{\frac{\pi}{2}}+\int_{\eta=\frac{\pi}{2}}^{0} \phi_{\xi=0}\, d\psi_{\xi=0}+\int_{\xi=0}^{\xi=a} \phi_{\eta=0}\, d\psi_{\eta=0}\right].$$

Substituting the values of (ϕ) and (ψ) from (6) and (1) respectively in the above, and simplifying the result, we have

$$T=\frac{\rho\, \omega^2 c^4}{32\,\pi}\left[2 \sinh \eta\, \alpha+8 \sum_{m=1}^{\infty} \frac{Q_{2m}}{4\, m_2-1} \tanh 2\, m\, \alpha\right.$$

$$+\sum_{m=1}^{\infty} \frac{Q_{2m+1}}{\sinh (2\, m+1)\, 2\, \alpha}\left\{\frac{\cosh (2\, m+1)\, 2\, \alpha-2 \cosh 2\, \alpha}{m\,(m+1)}\right.$$

$$\left.\left.-\cosh 2\, \alpha\left(\frac{\cosh 4\,(m+1)\, \alpha}{m+1}-\frac{\cosh 4\, m\, \alpha}{m}\right)\right\}\right],$$

where

$$Q_m=\left(\frac{1}{m+1}+\frac{1}{m-1}-\frac{2}{m}\right). \tag{7}$$

5. The corresponding expressions for ϕ, ψ and T for liquid motion inside a rotating circular quadrant can be obtained easily from (2), (6) and (7). For this, diminish c and increase α and ξ indefinitely such that

$$2\, c \cosh \alpha=2\, c\, \sinh \alpha=c\, e^{\alpha}=2\, a \text{ (say)},$$

$$c\, e^{\xi}=2\, r, \text{ where } r \text{ is the radius vector},$$

in which case, the elliptic quadrant becomes a circular quadrant of radius (a), and r, η are polar co-ordinates of any point referred to a bounding radius as initial line.

Observing that

$$\frac{c^2 \sinh 2\, m\,(\xi-\alpha)}{\sinh 2\, m\, \alpha}=0,\ c^2\, \frac{\sinh 2\, m\, \xi}{\sinh 2\, m\, \alpha}=0, \text{ etc.},$$

we have after a slight simplification,

$$\psi=\frac{\omega a^2}{4}\left[-\frac{4}{\pi}\left\{2 \frac{r^2}{a^2}\left(\eta-\frac{\pi}{4}\right) \cos 2\, \eta+2 \log\left(\frac{r}{a}\right). \frac{r^2}{a^2} \sin 2\, \eta\right\}\right.$$

$$\left.+\frac{6}{\pi}\, \frac{r^2}{a^2} \sin 2\, \eta-\frac{4}{\pi} \sum_{m=1}^{\infty} Q_{2m+1}\left(\frac{r^2}{a^2}\right)^{2\,m+1} \sin 2\,(2\, m+1)\, \eta\right] \tag{8}$$

and

$$\phi=\frac{\omega a^2}{4}\left[-\frac{4}{\pi}\left\{-2\frac{r^2}{a^2}\left(\eta-\frac{\pi}{4}\right)\sin 2\eta+2\log\left(\frac{r}{a}\right).\frac{r^2}{a^2}\cos 2\eta\right\}-\right.$$

$$\left.+\frac{6}{\pi}\frac{r^2}{a^2}\cos 2\eta-\frac{4}{\pi}\sum_{m=1}^{\infty}Q_{2m+1}\left(\frac{r^2}{a^2}\right)^{2m+1}\cos 2(2m+1)\eta\right). \quad (9)$$

Referring to bisector of the quadrant as initial line and putting

$\eta=\theta+\dfrac{\pi}{4}$, we have from (8) and (9),

$$(\psi-i\phi)=\frac{2\omega a^2}{\pi}\left[-\frac{r^2}{a^2}\left(\log\frac{r}{a}+i\theta\right)\left(\cos 2\theta+i\sin 2\theta\right)\right.$$

$$+\frac{3}{4}\frac{r^2}{a^2}(\cos 2\theta+i\sin 2\theta)+\frac{1}{2}\sum_{m=1}^{\infty}(-1)^{m+1}Q_{2m+1}\left(\frac{r^2}{a^2}\right)^{2m+1}$$

$$\left..\{\cos(2m+1)2\theta+i\sin(2m+1)2\theta\}\right]$$

$$=\frac{2\omega a^2}{\pi}\left[-x^2\log x+\frac{3}{4}x^2+\frac{1}{2}\sum_{m=1}^{\infty}(-1)^{m+1}Q_{2m+1}x^{2(2m+1)}\right],$$

where $x=\dfrac{r}{a}e^{i\theta}$.

Now, $\dfrac{3}{4}x^2+\dfrac{1}{2}\displaystyle\sum_{m=1}^{\infty}(-1)^{m+1}Q_{2m+1}x^{2(2m+1)}$

$$=x^2-\frac{x^2}{4}+\sum_{m=1}^{\infty}(-1)^m\frac{(x^2)^{2m+1}}{2m+1}+\frac{x^2}{4}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(x^4)^m}{m}$$

$$+\frac{1}{4x^2}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(x^4)^{m+1}}{m+1}$$

$$=\left\{x^2+\sum_{m=1}^{\infty}(-1)^m\frac{(x^2)^{2m+1}}{2m+1}\right\}+\frac{x^2}{4}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(x^4)^m}{m}$$

$$-\frac{1}{4x^2}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(x^4)^m}{m}$$

$$=\tan^{-1}x^2+\frac{1}{4}\left(x^2-\frac{1}{x^2}\right)\log(1+x^2).$$

Hence

$$\psi - i\phi = \frac{2\omega a^2}{\pi}\left[-x^2\log x + \tan^{-1}x^2 + \frac{1}{4}\left(x^2 - \frac{1}{x^2}\right)\log(1+x^4)\right]. \quad (10)$$

Also

$$T = \frac{\rho\omega^2 a^4}{2\pi}\left[1 - \sum_{m=1}^{\infty}\frac{Q_{2m+1}}{2(m+1)}\right]$$

$$= \frac{\rho\omega^2 a^4}{2\pi}\left[1 - \sum_{m=1}^{\infty}\left\{\frac{1}{4(m+1)^2} + \frac{1}{4m(m+1)} - \frac{2}{(2m+1)(2m+2)}\right\}\right].$$

Evidently,

$$\sum_{m=1}^{\infty}\frac{1}{4(m+1)^2} = \frac{1}{4}\left((+\frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots \infty) - 1\right) = \frac{1}{4}\left(\frac{\pi^2}{6} - 1\right)$$

$$\sum_{m=1}^{\infty}\frac{1}{4(m+1)\,m} = \frac{1}{4} \text{ and } \sum_{m=1}^{\infty}\frac{1}{(2m+1)(2m+2)} = \log 2 - \frac{1}{2}.$$

Therefore

$$T = \frac{\rho\omega^2 a^4}{2\pi}\left[1 - \frac{1}{4}\left(\frac{\pi^2}{6} - 1\right) - \frac{1}{4} + 2\log 2 - 1\right]$$

$$= \frac{\rho\omega^2 a^4}{\pi}\left(\log 2 - \frac{\pi^2}{48}\right). \quad (11)$$

The expressions (10) and (11) are identical with those obtained by Sir A. G. Greenhill(1).

6. A few alternative expressions for ϕ and ψ may be obtained by the method indicated by me in a previous paper(2). The expressions for ϕ and ψ thus found are as follows:

$$\phi = \frac{\omega c^2}{4}\left[-\frac{4}{\pi}\left\{(\xi - \alpha)\cosh 2\xi \cos 2\eta - \left(\eta - \frac{\pi}{2}\right)\sin 2\xi \sin 2\eta\right.\right.$$

$$\left. - \frac{3}{4}\sinh 2\xi \cos 2\eta - \frac{\pi}{4}\tanh\alpha \cosh 2\xi\, 2\eta\right\}$$

$$+ \sum_{m=1}^{m=\infty} M\{\cosh 2(2m+1)(\xi - \alpha) - \cosh 2\alpha \cosh 2(2m+1)\xi\}$$

$$\times \cos 2(2m+1)\eta$$

$$\left. + \sum_{m=0}^{\infty} N.\cosh\frac{(2m+1)\pi}{\alpha}\left(\eta - \frac{\pi}{4}\right)\cos(2m+1)\frac{\pi\xi}{\alpha}\right] \quad (12)$$

(1) Greenhill, Mess. Math. Vol. 8 ibid.

(2) "Liquid motion inside certain rotating curvilinea rectangles" Bull. Cal. Math. soc. Vol 9, 7, 1920.

and
$$\psi=\frac{\omega c^2}{4}\left[-\frac{4}{\pi}\left\{(\xi-\alpha)\sinh 2\xi \sin 2\eta+\left(\eta-\frac{\pi}{2}\right)\cosh 2\xi\cos 2\eta\right.\right.$$
$$\left.-\frac{3}{4}\cosh 2\xi\sin 2\eta+\frac{\pi}{4}\tanh\alpha\sinh 2\xi\cos 2\eta\right\}$$
$$+\sum_{m=1}^{\infty} M\{\sinh 2(2m+1)(\xi-\alpha)$$
$$-\cosh 2\alpha\sinh 2(2m+1)\xi\}\sin 2(2m+1)\eta$$
$$\left.-\sum_{m=0}^{\infty} N\sinh\frac{(2m+1)\pi}{\alpha}\left(\eta-\frac{\pi}{4}\right)\sin\frac{(2m+1)\pi\xi}{\alpha}\right], \tag{13}$$

where
$$M=\frac{1}{\pi m(2m+1)(m+1)\sinh 2(2m+1)\alpha}, \tag{14}$$
$$N=\frac{16\alpha^2}{\pi(2m+1)\{(2m+1)^2\pi^2+4\alpha^2\}\sinh(2m+1)\pi^2/4\alpha}. \tag{15}$$

The results (6) and (2) can be easily obtained from (12) and (13) by the help of a few analytical theorems proved by me in the paper cited above, viz:

$$\frac{\pi}{16}\frac{\cosh 2\phi\sin 2\theta}{\cosh 2\epsilon_1\sin 2\epsilon_2}-\frac{\pi\theta}{16\epsilon_2}$$
$$+\sum_{2m+1}^{\infty}\frac{(-1)^m}{2m+1}\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_2}}{\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}+4\right\}\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi$$
$$+\sum_{m=1}^{\infty}\frac{(-1)^m}{2m\left(\frac{m^2\pi^2}{\epsilon_2^2}-4\right)}\frac{\cosh\frac{m\pi\phi}{\epsilon_2}}{\cosh\frac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi\theta}{\epsilon_2}=0, \tag{16}$$

$$\frac{\pi}{16}\frac{\sinh 2\phi\cos 2\theta}{\sinh 2\epsilon_1\cos 2\epsilon_2}-\frac{\pi\phi}{16\epsilon_1}$$
$$+\sum_{0}^{\infty}\frac{(-1)^m}{(2m+1)\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_2^2}-4\right\}}\frac{\sinh\frac{(2m+1)\pi\phi}{2\epsilon_2}}{\sinh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\pi\theta}{2\epsilon_2}$$

$$+\sum_{1}^{\infty} \frac{(-1)^{m-1}}{2m\left\{\frac{m^2\pi^2}{\epsilon_1^2}+4\right\}} \frac{\cosh \frac{m\pi\theta}{\epsilon_1}}{\cosh \frac{m\pi}{\epsilon_1}\epsilon_2} \sin \frac{m\pi}{\epsilon_1}\phi = 0 \tag{17}$$

and

$$\frac{\pi}{16}\left(1-\frac{\cosh 2\phi \cos 2\theta}{\cosh 2\epsilon_1 \cos 2\epsilon_2}\right)$$

$$=\sum_{m=0}^{\infty}\left\{\frac{(-1)^{m+1}}{(2m+1)\left(\frac{(2m+1)^2\pi^2}{4\epsilon_2^2}-4\right)} \frac{\cosh \frac{(2m+1)\pi}{2\epsilon_2}\phi}{\cosh \frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}} \cos \frac{(2m+1)\pi\theta}{2\epsilon_2}\right.$$

$$\left.+\frac{(-1)^m}{(2m+1)\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}+4\right\}} \frac{\cosh \frac{(2m+1)\pi\theta}{2\epsilon_1}}{\cosh \frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}} \cos \frac{(2m+1)\pi\phi}{2\epsilon_1}\right\} \tag{18}$$

where $$-\epsilon_1<\phi<\epsilon_1,\ -\epsilon_2<\theta<\epsilon_2.$$

Putting $\xi-\frac{\alpha}{4}$, $\eta-\frac{\pi}{4}$, $\frac{\alpha}{2}$, $\frac{\pi}{4}$ for ϕ, θ_2, ϵ_1, ϵ_2 respectively in (16), we have

$$0=\frac{\pi}{16}\left[\tan \mathrm{h}\,\alpha \sin \mathrm{h}\, 2\xi \cos 2\eta - \cos \mathrm{h}\, 2\xi \cos 2\eta - \frac{4}{\pi}\left(\eta-\frac{\pi}{4}\right)\right.$$

$$+\sum_{m=0}^{\infty}\frac{16}{\pi(2m+1)} \frac{\sinh (2m+1)\frac{\pi}{2}\left(\eta-\frac{\pi}{4}\right)}{\left\{\frac{(2m+1)^2\pi^2}{\alpha^2}+4\right\}\sinh \frac{(2m+1)\pi^2}{4\alpha}} \sin (2m+1)\frac{\pi\xi}{\alpha}$$

$$\left.+\sum_{m=1}^{\infty}\frac{2}{\pi m(4m^2-1)} \frac{\cosh 4m\left(\xi-\frac{\alpha}{2}\right)}{\cosh 2m\alpha} \sin 4m\eta\right].$$

Eliminating the infinite series $\sum_{m=0}^{\infty} \frac{16}{\pi(2m+1)\left\{\frac{(2m+1)^2\pi^2}{\alpha^2}+4\right\}}$

$$\cdot \sinh\,(2\,m+1)\frac{\pi}{\alpha}\left(\eta-\frac{\pi}{4}\right)\frac{\sin\,(2\,m+1)\,\dfrac{\pi\xi}{\alpha}}{\sinh\dfrac{(2\,m+1)\,\pi^2}{4\,\alpha}}\Bigg\}$$

from the above identity and (13), we have

$$\psi=\frac{\omega c^2}{4}\Bigg[-\frac{4}{\pi}\Bigg\{(\xi-\alpha)\sinh 2\,\xi\sin 2\,\eta+\left(\eta-\frac{\pi}{4}\right)\cosh 2\,\xi\cos 2\,\eta+\left(\eta-\frac{\pi}{4}\right)\Bigg\}$$

$$+\frac{3}{\pi}\cosh 2\,\xi\sin 2\,\eta+\sum_{m=1}^{\infty}\frac{2}{\pi\,m\,(4\,m^2-1)}\,\frac{\cosh 4\,m\left(\xi-\dfrac{\alpha}{2}\right)}{\cosh 2\,m\,\alpha}\sin 4\,m\,\eta$$

$$+\sum_{m=1}^{\infty}M\,\{\sinh 2\,(2\,m+1)\,(\xi-\alpha)-\cosh 2\,\alpha\sinh 2\,(2\,m+1)\,\xi\}$$

$$\cdot \sin 2\,(2\,m+1)\,\eta\Bigg].$$

This expression for ψ may be readily identified with (2) by (3), (4) and (5). The corresponding alternative expression for ϕ can be written down as before.

A third alternative expression for ψ may be deduced in a similar way from (13) by eliminating the infinite series

$$\sum_{m=1}^{\infty}M\,\{\sinh 2\,(2\,m+1)\,(\varepsilon-\alpha)-\cosh 2\,\alpha\sinh 2\,(2\,m+1)\,\eta\}$$

$$\cdot \sin 2\,(2\,m+1)\,\eta$$

with the help of the identities (17) and (18), which reduce to an indeterminate form $\infty-\infty$ when $\dfrac{\pi}{4}$ is substituted for ϵ_2, but which can be easily evaluated by putting $2\,\varepsilon_2=\dfrac{\pi}{2}+x$, where Lt. $x=0$. The corresponding expression for ϕ can be written down as before.

Motion under a Central Force with Infinitesimal Transformations,

by

TORANOSUKE IWATSUKI, Yamaguchi.

In this paper we consider the infinitesimal transformations which transform a system of curves of motion under a central force into itself, and then find the conditions of integrability of equations of motion and some other properties.

We will call our motion "the central motion," and the curves described by the motion, "the central curves."

Take the origin of the rectangular coordinates at the point in the plane (x, y), in which the lines of force intersect, then the central motion satisfies the following equation:

$$x\dot{y}-y\dot{x}=k, \tag{1}$$

where k is an arbitrary constant.

I. The infinitesimal transformation by which $x\dot{y}-y\dot{x}=k$ remains unchanged.

As our transformation depends only upon the coordinates x, y and the time t, it must be expressed in the form

$$Uf\equiv\xi(x, y, t)\frac{\partial f}{\partial x}+\eta(x, y, t)\frac{\partial f}{\partial y}+\zeta(x, y, t)\frac{\partial f}{\partial t}.$$

And if we put

$$\Omega\equiv x\dot{y}-y\dot{x},$$

then in order that $\Omega=0$ may be invariant by the transformation Uf, it is necessary and sufficient that

$$\dot{U}\Omega\equiv\lambda.\Omega, \tag{2}$$

where $$\dot{U}f\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t}+(\dot{\xi}-\dot{x}\dot{\zeta})\frac{\partial f}{\partial \dot{x}}+(\dot{\eta}-\dot{y}\dot{\zeta})\frac{\partial f}{\partial \dot{y}},$$

and λ is, in general, a function of x, y, t, $\dot{x}$, $\dot{y}$[1].

Rewriting (2), we get actually

$$\left(x\frac{\partial\eta}{\partial x}-y\frac{\partial\xi}{\partial x}-\eta\right)\dot{x}+\left(x\frac{\partial\eta}{\partial y}-y\frac{\partial\xi}{\partial y}+\xi\right)\dot{y}+\left(x\frac{\partial\eta}{\partial t}-y\frac{\partial\xi}{\partial t}\right)$$
$$-k\left(\dot{x}\frac{\partial\zeta}{\partial x}+\dot{y}\frac{\partial\zeta}{\partial y}+\frac{\partial\zeta}{\partial t}\right)\equiv\lambda.\,(x\dot{y}-y\dot{x}-k).$$

But the above must be identical with respect to all values of $\dot{x}$ and $\dot{y}$, so we have

$$x\frac{\partial\eta}{\partial x}-y\frac{\partial\xi}{\partial x}-\eta-k\frac{\partial\zeta}{\partial x}=-y\lambda,$$
$$x\frac{\partial\eta}{\partial y}-y\frac{\partial\xi}{\partial y}+\xi-k\frac{\partial\zeta}{\partial y}=x\lambda,$$
$$x\frac{\partial\eta}{\partial t}-y\frac{\partial\xi}{\partial t}-k\frac{\partial\zeta}{\partial t}=-k\lambda.$$

Rewriting the above,

$$\frac{\partial f}{\partial x}-2\eta=-y\lambda,$$
$$\frac{\partial f}{\partial y}+2\xi=x\lambda,$$
$$\frac{\partial f}{\partial t}=-k\lambda,$$

where $$f\equiv x\eta-y\xi-k\zeta.$$

And if we put $f\equiv-2\,\varphi$, then from the above four relations we can obtain ξ, η, ζ as follows:

$$\xi=\varphi_y+\frac{x}{k}\,\varphi_t,\ \eta=-\varphi_x+\frac{y}{k}\,\varphi_t,\ \zeta=\frac{1}{k}(2\,\varphi-x\varphi_x-y\varphi_y).$$

So we have

Theorem 1: In order that a central motion, whose equation is $x\dot{y}-y\dot{x}=k$, may be transformed into itself by the infinitesimal transformation, it is necessary and sufficient that it has the form

$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

(1) S. Lie, Vorlesungen ü. Differentialgleichungen, S. 277.

where $\xi=\varphi_y+\frac{x}{k}\varphi_t$, $\eta=-\varphi_x+\frac{y}{k}\varphi_t$, and $\zeta=\frac{1}{k}(2\varphi-x\varphi_x-y\varphi_y)$, in which φ is an arbitrary function of x, y, t.

As we can write also

$$Uf\equiv U_1f+\frac{1}{k}U_2f,$$

where

$$U_1f\equiv\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y},\quad U_2f\equiv x\varphi_t\frac{\partial f}{\partial x}+y\varphi_t\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

we know that the infinitesimal transformation for which

$$x\ddot{y}-y\ddot{x}=0$$

is invariant, is composed of two transformations U_1f and U_2f.

So we have

Theorem 2: If the infinitesimal transformation has an invariant central motion $x\ddot{y}-y\ddot{x}=0$, then it is composed of two infinitesimal transformations

$$U_1f\equiv\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y}\quad\text{and}\quad U_2f\equiv x\varphi_t\frac{\partial f}{\partial x}+y\varphi_t\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t}.$$

II. Integration of the equations of the general central motion.

We consider the integration of the equations of motion by means of the infinitesimal transformations.

The general equations of central motion are

$$\left.\begin{aligned}\ddot{x}&=x\,.\,F(x,\ y,\ t,\ \dot{x},\ \dot{y})\\ \ddot{y}&=y\,.\,F(x,\ y,\ t,\ \dot{x},\ \dot{y})\end{aligned}\right\}, \tag{3}$$

and they have

$$x\dot{y}-y\dot{x}=k$$

as the first integral; hence if there exists an infinitesimal transformation, which the equations of motion (3) admit of, then it must be of the form

$$Uf\equiv\left(\varphi_y+\frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x}+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t}.$$

Next, if we put $\qquad \Omega\equiv\ddot{x}-x\,.\,F$

and

$$\ddot{U}f \equiv \left(\varphi_y + \frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x} + \left(-\varphi_x + \frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y} + \zeta$$

$$+\left(\dot{\varphi}_y + \frac{\dot{x}}{k}\varphi_t + \frac{x}{k}\dot{\varphi}_t - \dot{x}\zeta\right)\frac{\partial f}{\partial \dot{x}} + \left(-\dot{\varphi}_x + \frac{\dot{y}}{k}\varphi_t + \frac{y}{k}\dot{\varphi}_t - \dot{y}\zeta\right)\frac{\partial f}{\partial \dot{y}}$$

$$+\left(\ddot{\varphi}_y + \frac{\ddot{x}}{k}\varphi_t + 2\frac{\dot{x}}{k}\dot{\varphi}_t - 2\ddot{x}\dot{\zeta} - \dot{x}\ddot{\zeta}\right)\frac{\partial f}{\partial \ddot{x}}$$

$$+\left(-\ddot{\varphi}_x + \frac{\ddot{y}}{k}\varphi_t + 2\frac{\dot{y}}{k}\dot{\varphi}_t - 2\ddot{y}\dot{\zeta} - \dot{y}\ddot{\zeta}\right)\frac{\partial f}{\partial \ddot{y}},$$

then in order that $\Omega = 0$ may admit of the infinitesimal transformation Uf, it is necessary and sufficient that

$$\ddot{U}\Omega = 0,$$

by virtue of $\Omega = 0$ (1).

So we have the condition

$$-x\left[\left(\varphi_y + \frac{x}{k}\varphi_t\right)\frac{\partial F}{\partial x} + \left(-\varphi_x + \frac{y}{k}\varphi_t\right)\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial t} + \left(\dot{\varphi}_y + \frac{\dot{x}}{k}\varphi_t + \frac{x}{k}\dot{\varphi}_t\right.\right.$$

$$\left.\left. - \dot{x}\dot{\zeta}\right)\frac{\partial F}{\partial \dot{x}} + \left(-\dot{\varphi}_x + \frac{\dot{y}}{k}\varphi_t + \frac{y}{k}\dot{\varphi}_t - \dot{y}\dot{\zeta}\right)\frac{\partial F}{\partial \dot{y}}\right]$$

$$-\left(\varphi_y + \frac{x}{k}\ddot{\varphi}_t\right)F + \ddot{\varphi}_y + \frac{x}{k}\varphi_t F + 2\frac{\dot{x}}{k}\dot{\varphi}_t + \frac{x}{k}\ddot{\varphi}_t - 2xF\dot{\zeta} - \dot{x}\ddot{\zeta} = 0. \quad (a)$$

Similarly, in order that $\ddot{y} = yF$ may admit of the same transformation Uf, we have the condition

$$-y\left[\left(\varphi_y + \frac{x}{k}\varphi_t\right)\frac{\partial F}{\partial x} + \left(-\varphi_x + \frac{y}{k}\varphi_t\right)\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial t}\right.$$

$$\left. + \left(\dot{\varphi}_y + \frac{\dot{x}}{k}\varphi_t + \frac{x}{k}\dot{\varphi}_t - \dot{x}\dot{\zeta}\right)\frac{\partial F}{\partial \dot{x}} + \left(-\dot{\varphi}_x + \frac{\dot{y}}{k}\varphi_t + \frac{y}{k}\dot{\varphi}_t - \dot{y}\dot{\zeta}\right)\frac{\partial F}{\partial \dot{y}}\right]$$

$$-\left(-\varphi_x + \frac{y}{k}\varphi_t\right)F - \ddot{\varphi}_x + \frac{y}{k}\varphi_t F + 2\frac{\dot{y}}{k}\dot{\varphi}_t + \frac{y}{k}\ddot{\varphi}_t - 2yF\dot{\zeta} - \dot{y}\ddot{\zeta} = 0. \quad (b)$$

Multiplying (a) and (b) by y and x respectively, and subtracting we have

(1) S. Lie, Vorlesungen ü. Differentialgleichungen, S. 362.

$$-(x\varphi_x+y\varphi_y)\,F+x\ddot{\varphi}_x+y\ddot{\varphi}_y-2\dot{\varphi}_t+k\ddot{\zeta}=0 \quad (c);$$

but differentiating the relation $k\zeta=2\varphi-x\varphi_x-y\varphi_y$ with respect to t and using the equations of motion (3), we have

$$k\ddot{\zeta}=(x\varphi_x+y\varphi_y)\,F-x\ddot{\varphi}_x-y\ddot{\varphi}_y+2\,\dot{\varphi}_t;$$

and then substituting the value of ζ above obtained into (c), we may easily see that (c) is an identity.

Hence (a) and (b) are not independent, but are to be connected by a linear relation. Therefore, multiplying (a) and (b) by $\dot{y}$ and $\dot{x}$ respectively, and subtracting we have

$$(k\varphi_y+x\varphi_t)\frac{\partial F}{\partial x}+(-k\varphi_x+y\varphi_t)\frac{\partial F}{\partial y}+k\zeta\frac{\partial F}{\partial t}+(k\dot{\varphi}_y+\dot{x}\varphi_t+x\dot{\varphi}_t-k\dot{x}\dot{\zeta})\frac{\partial F}{\partial \dot{x}}$$

$$+(-k\dot{\varphi}_x+\dot{y}\varphi_t+y\dot{\varphi}_t-k\dot{y}\dot{\zeta})\frac{\partial F}{\partial \dot{y}}=(2\,\varphi_t-3\,k\dot{\zeta})\,F+\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^3\varphi,$$

which is the relation that stands for both (a) and (b).

So we have

Theorem 3: In order that the equations of motion

$$\ddot{x}=x\,.\,F(x,y,t,\dot{x},\dot{y}),$$

$$\ddot{y}=y\,.\,F(x,y,t,\dot{x},\dot{y}),$$

the first integral being $x\dot{y}-y\dot{x}=k$, may admit of an infinitesimal transformation, it is necessary and sufficient that F is the solution of a partial differential equation

$$(k\varphi_y+x\varphi_t)\frac{\partial F}{\partial x}+(-k\varphi_x+y\varphi_t)\frac{\partial F}{\partial y}+k\dot{\zeta}\frac{\partial F}{\partial t}+(k\dot{\varphi}_y+\dot{x}\varphi_t+x\dot{\varphi}_t-k\dot{x}\dot{\zeta})\frac{\partial F}{\partial \dot{x}}$$

$$+(-k\dot{\varphi}_x+\dot{y}\varphi_t+y\dot{\varphi}_t-k\dot{y}\dot{\zeta})\frac{\partial F}{\partial \dot{y}}=(2\,\varphi_t-3k\dot{\zeta})\,F+\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^3\varphi,$$

where φ is an arbitrary function of x, y, t. And the infinitesimal transformation is of the form

$$\left(\varphi_y+\frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x}+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t}.$$

We shall further consider the case in which the equations of motion admit of two infinitesimal transformations having the forms

$$U_1 f \equiv \left(\varphi_y + \frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x} + \left(-\varphi_x + \frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial t},$$

$$U_2 f \equiv \left(\psi_y + \frac{x}{k}\psi_t\right)\frac{\partial f}{\partial x} + \left(-\psi_x + \frac{y}{k}\psi_t\right)\frac{\partial f}{\partial y} + \bar{\zeta}\frac{\partial f}{\partial t},$$

where

$$k\zeta \equiv 2\varphi - x\varphi_x - y\varphi_y, \quad k\bar{\zeta} \equiv 2\psi - x\psi_x - y\psi_y.$$

In this case, F must be, by Theorem 3, a common solution of the following two partial differential equations:

$$AF \equiv (k\varphi_y + x\varphi_t)\frac{\partial F}{\partial x} + (-k\varphi_x + y\varphi_t)\frac{\partial F}{\partial y} + k\zeta\frac{\partial F}{\partial t} + (k\dot{\varphi}_y + \dot{x}\varphi_t$$

$$+ x\dot{\varphi}_t - k\dot{x}\dot{\zeta})\frac{\partial F}{\partial \dot{x}} + (-k\dot{\varphi}_x + \dot{y}\varphi_t + y\dot{\varphi}_t - k\dot{y}\dot{\zeta})\frac{\partial F}{\partial \dot{y}} - (2\varphi_t - 3k\dot{\zeta})F$$

$$-\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)^3\varphi = 0,$$

$$BF \equiv (k\psi_y + x\psi_t)\frac{\partial F}{\partial x} + (-k\psi_x + y\psi_t)\frac{\partial F}{\partial y} + k\bar{\zeta}\frac{\partial F}{\partial t} + (k\dot{\psi}_y + \dot{x}\psi_t + x\dot{\psi}_t$$

$$-k\dot{x}\dot{\bar{\zeta}})\frac{\partial F}{\partial \dot{x}} + (-k\dot{\psi}_x + \dot{y}\psi_t + y\dot{\psi}_t - k\dot{y}\dot{\bar{\zeta}})\frac{\partial F}{\partial \dot{y}} - (2\psi_t - 3k\dot{\bar{\zeta}})F$$

$$-\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)^3\psi = 0.$$

If we put

$$\bar{A}F \equiv (k\varphi_y + x\varphi_t)\frac{\partial F}{\partial x} + (-k\varphi_x + y\varphi_t)\frac{\partial F}{\partial y} + k\zeta\frac{\partial F}{\partial t}$$

$$+ (k\dot{\varphi}_y + \dot{x}\varphi_t + x\dot{\varphi}_t - k\dot{x}\dot{\zeta})\frac{\partial F}{\partial \dot{x}} + (-k\dot{\varphi}_x + \dot{y}\varphi_t + y\dot{\varphi}_t$$

$$-k\dot{y}\dot{\zeta})\frac{\partial F}{\partial \dot{y}} + (3k\dot{\zeta} - 2\varphi_t)F,$$

and

$$\bar{B}F \equiv (k\psi_y + x\psi_t)\frac{\partial F}{\partial x} + (-k\psi_x + y\psi_t)\frac{\partial F}{\partial y} + k\zeta\frac{\partial F}{\partial t}$$

$$+ (k\dot{\psi}_y + \dot{x}\psi_t + x\dot{\psi}_t - k\dot{x}\dot{\zeta})\frac{\partial F}{\partial \dot{x}}$$

$$+ (-k\dot{\psi}_x + \dot{y}\psi_t + y\dot{\psi}_t - k\dot{y}\dot{\zeta})\frac{\partial F}{\partial \dot{y}} + (3\,x\dot{\zeta} - 2\,\psi_t)\,F,$$

then the necessary and sufficient condition, that $AF=0$ and $BF=0$ may have a common solution, is

$$\bar{A}B - \bar{B}A \equiv \lambda_1\,.\,A + \lambda_2\,.\,B, \tag{4}$$

where λ_1 and λ_2 are, in general, functions of x, y, t, $\dot{x}$, $\dot{y}$.

In our case,

$$\bar{A}B - \bar{B}A = (k\dot{\chi}_y + x\chi_t)\frac{\partial F}{\partial x} + (-k\dot{\chi}_x + y\chi_t)\frac{\partial F}{\partial y} + k\rho\frac{\partial F}{\partial t}$$

$$+ (k\dot{\chi}_y + \dot{x}\chi_t + x\dot{\chi}_t - k\dot{x}\dot{\rho})\frac{\partial F}{\partial \dot{x}}$$

$$+ (-k\dot{\chi}_x + \dot{y}\chi_t + x\dot{\chi}_t - x\dot{y}\dot{\rho})\frac{\partial F}{\partial \dot{y}} + (3\,k\dot{\rho} - 2\chi_t)\,F$$

$$-\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)^3 \chi,$$

where

$$\chi \equiv \varphi_y\psi_x - \varphi_x\psi_y + \frac{2}{k}(\varphi\psi_t - \psi\varphi_t) - \frac{x}{k}(\varphi_x\psi_t - \varphi_t\psi_x) - \frac{y}{k}(\varphi_y\psi_t - \varphi_t\psi_y),$$

and

$$\rho \equiv \frac{1}{k}(2\,\chi - x\chi_x - y\chi_y).$$

Hence in order that the condition (4) may hold, we must have

$$\chi = \alpha\varphi + \beta\psi,$$

where α, β are some constants.

We can also easily see that

$$(U_1,\ U_2)\equiv U_1(U_2f)-U_2(U_1f)$$

$$=\left(\chi_y+\frac{x}{k}\chi_t\right)\frac{\partial f}{\partial x}+\left(-\chi_x+\frac{y}{k}\chi_t\right)\frac{\partial f}{\partial y}+\rho\frac{\partial f}{\partial t},$$

and then substituting the value of χ, we have

$$(U_1,\ U_2)\equiv\alpha\,.\,U_1f+\beta\,.\,U_2f;$$

it shows that the two infinitesimal transformations U_1f and U_2f form the "2 gliedrige Gruppe"(1).

Conversely, if U_1f and U_2f form the "2 gliedrige Gruppe," that is,

$$(U_1,\ U_2)\equiv\alpha\,.\,U_1f+\beta\,.\,U_2f,$$

in which α and β are any constants, then we have

$$\chi\equiv\alpha\varphi+\beta\psi,$$

so that $AF=0$ and $BF=0$ may admit of a common solution. Hence we have

Theorem 4: In order that the equations of a central motion may admit of two infinitesimal transformations, it is necessary and sufficient that they form the "2 gliedrige Gruppe."

And also we have

Theorem 5: If the equations of a central motion

$$\ddot{x}=x\,.\,F\ ,\ \ddot{y}=y\,.\,F$$

admit of two infinitesimal transformations

$$U_1f\equiv\left(\varphi_y+\frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x}+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

$$U_2f\equiv\left(\psi_y+\frac{x}{k}\psi_t\right)\frac{\partial f}{\partial x}+\left(-\psi_x+\frac{y}{k}\psi_t\right)\frac{\partial f}{\partial y}+\bar{\zeta}\frac{\partial f}{\partial t},$$

then there exists a relation between φ and ψ:

$$\varphi_y\psi_x-\psi_y\varphi_x+\frac{2}{k}(\varphi\psi_t-\psi\varphi_t)-\frac{x}{k}(\varphi_x\psi_t-\psi_x\varphi_t)-\frac{y}{k}(\varphi_y\psi_t-\psi_y\psi_t)=\alpha\psi+\beta\varphi;$$

and F is a common solution of $AF=0$ and $BF=0$.

Before we proceed in our considerations, we must prove some properties concerning the expression

(1) S. Lie, l. c., S. 406.

$$\varphi_y\psi_x-\psi_y\varphi_x+\frac{2}{k}(\varphi\psi_t-\psi\varphi_t)-\frac{x}{k}(\varphi_x\psi_t-\psi_x\varphi_t)-\frac{y}{k}(\varphi_y\psi_t-\psi_y\varphi_t).$$

If we put for the above expression $[\psi, \varphi]$, then we see at once that

$$[\varphi, \varphi]\equiv 0, \ [\psi, \varphi]\equiv-[\varphi, \psi], \tag{5}$$

and also that

$$\left[\frac{\psi}{\varphi}, \varphi\right]=\frac{1}{\varphi^2}\{[\psi, \varphi]\,\varphi-[\varphi, \varphi]\,\psi\}$$

$$=\frac{1}{\varphi}[\psi, \varphi],$$

but on the other hand,

$$\left[\frac{\psi}{\varphi}, \varphi\right]\equiv\left(\varphi_y+\frac{x}{k}\varphi_t\right)\left(\frac{\psi}{\varphi}\right)_x+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\left(\frac{\psi}{\varphi}\right)_y+\zeta\left(\frac{\psi}{\varphi}\right)_t,$$

hence we have

$$U\frac{\psi}{\varphi}=\frac{1}{\varphi}[\psi, \varphi]. \tag{6}$$

So we have the

Lemma 1: If functions φ and ψ satisfy the relation

$$[\psi, \varphi]=\alpha\psi+\beta\varphi,$$

then we have

$$U\frac{\psi}{\varphi}=\alpha\frac{\psi}{\varphi}+\beta.$$

Next, if $[\lambda, \varphi]=\alpha_1\lambda+\beta_1\varphi$ and $[\mu, \varphi]=\alpha_2\mu+\beta_2\varphi$ coexist, then the two differential equations have a common solution φ, and accordingly there exists an identical relation between λ, μ, φ.

In our case, finding the condition of coexistency, we have

$$[[\lambda, \mu], \varphi]\equiv(\alpha_1+\alpha_2)\,[\lambda, \mu]+\alpha_1\beta_2\lambda-\alpha_2\beta_1\mu\,; \tag{7}$$

hence if we assume that $[\lambda, \mu]=\alpha\lambda+\beta\mu$, then (7) becomes

$$[(\alpha\lambda+\beta\mu), \varphi]\equiv(\alpha_1+\alpha_2)\,(\alpha\lambda+\beta\mu)+\alpha_1\beta_2\lambda-\alpha_2\beta_1\mu,$$

or

$$\alpha(\alpha_1\lambda+\beta_1\varphi)+\beta(\alpha_2\mu+\beta_2\varphi)\equiv(\alpha_1+\alpha_2)(\alpha\lambda+\beta\mu)+\alpha_1\beta_2\lambda-\alpha_2\beta_1\mu.$$

But the above being an identity for all values of λ, μ, φ, the coefficients of λ, μ, φ must vanish; that is,

$$\alpha\alpha_2+\beta_2\alpha_1=0,\ \beta\alpha_1-\beta_1\alpha_2=0,\ \alpha\beta_1+\beta\beta_2=0. \qquad (8)$$

Hence we have the

Lemma 2: In order that $[\lambda, \varphi]=\alpha_1\lambda+\beta_1\varphi$ and $[\mu, \varphi]=\alpha_2\mu+\beta_2\varphi$ may coexist, it is sufficient that

$$[\lambda, \mu]=\alpha\lambda+\beta\mu,$$

whence follow the relations

$$\alpha\alpha_2+\beta_2\alpha_1=0,\ \beta\alpha_1-\beta_1\alpha_2=0,\ \alpha\beta_1+\beta\beta_2=0.$$

From the above, we also have the

Lemma 3: If φ is any function of x, y, t, then we can obtain the other functions λ, μ and ν which satisfy the following six relations:

$$[\lambda, \varphi]=0,\ [\mu, \varphi]=0,\ [\nu, \varphi]=\alpha\nu+\beta\varphi,$$

$$[\lambda, \nu]=0,\ [\mu, \nu]=0,\ [\lambda, \mu]=\alpha_1\lambda+\beta_1\mu,$$

where $\alpha, \beta, \alpha_1, \beta_1$ are any constants and both α and β, or α_1 and β_1 do not vanish.

Because in any two of the six equations containing the same letter in the left sides, the coefficients in the rights satisfy the relation (8).

Now, using the above lemmas, let us consider the integration of the equations of motion.

Let u, v, w be the functions of x, y, t independent of each other, and change the variables in

$$xdy-ydx=kdt$$

such as

$$u=u(x, y, t),\ v=v(x, y, t),\ w=w(x, y, t); \qquad (9)$$

then it becomes

$$\{k(w_x v_y-w_y v_x)-x(w_t v_x-w_x v_t)-y(w_t v_y-w_y v_t)\}\, du$$

$$+\{k(u_x w_y-u_y w_x)-x(u_t w_x-u_x w_t)-y(u_t w_y-u_y w_t)\}\, dv$$

$$+\{k(v_x u_y - v_y u_x) - x(v_t u_x - v_x u_t) - y(v_t w_y - v_y u_t)\}\, dw = 0. \quad (10)$$

Again put

$$u = \frac{\lambda}{\varphi},\quad v = \frac{\mu}{\varphi},\quad w = \frac{\nu}{\varphi},$$

then

$$w_x v_y - w_y v_x = \frac{1}{\varphi^3}\{(\nu_x \mu_y - \mu_x \nu_y)\varphi - (\varphi_y \nu_x - \nu_y \varphi_x)\mu + \nu(\mu_x \varphi_y - \mu_y \varphi_x)\},$$

$$w_t v_x - w_x v_t = \frac{1}{\varphi^3}\{(\nu_t \mu_x - \mu_t \nu_x)\varphi - (\varphi_x \nu_t - \nu_x \varphi_t)\mu + \nu(\mu_t \varphi_x - \mu_x \varphi_t)\},$$

$$w_t v_y - w_y v_t = \frac{1}{\varphi^3}\{(\nu_t \mu_y - \mu_t \nu_y)\varphi - (\varphi_y \nu_t - \nu_y \varphi_t)\mu + \nu(\mu_t \varphi_y - \mu_y \varphi_t)\};$$

but as we know

$$0 \equiv \frac{1}{\varphi^3}\{(\nu_t \mu - \mu_t \nu)\varphi - (\varphi\nu_t - \nu\varphi_t)\mu + \nu(\mu_t \varphi - \mu\varphi_t)\},$$

we have, multiplying the above four relations by $k, -x, -y$, 1 respectively, and adding,

$$k(w_x v_y - w_y v_x) - x(w_t v_x - w_x v_t) - y(w_t v_y - w_y v_t)$$
$$= \frac{k}{\varphi^3}\{[\nu, \mu]\varphi - [\nu, \varphi]\mu + [\mu, \varphi]\nu\}.$$

Hence (10) becomes

$$\{[\nu, \mu]\varphi - [\nu, \varphi]\mu + [\mu, \varphi]\nu\}\, du + \{[\lambda, \nu]\varphi - [\lambda, \varphi]\nu + [\nu, \varphi]\lambda\}\, dv$$
$$+ \{[\mu, \lambda]\varphi - [\mu, \varphi]\lambda + [\lambda, \varphi]\mu\}\, dw = 0. \quad (11)$$

Especially, if we take λ, μ, ν as in Lemma 3, then (11) becomes

$$(\lambda\, dv - \mu\, du)(\alpha\nu + \beta\varphi) = \varphi\,(\alpha_1 \lambda + \beta_1 \mu)\, dw,$$

or dividing by φ^2, we have

$$(u\, dv - v\, du)(\alpha\, w + \beta) = (\alpha_1 u + \beta_1 v)\, dw. \quad (12)$$

Applying the same change of variables into the expression of the infinitesimal transformation Uf, we have

$$Uf \equiv U\frac{\lambda}{\varphi} \cdot \frac{\partial f}{\partial u} + U\frac{\mu}{\varphi} \cdot \frac{\partial f}{\partial v} + U\frac{\nu}{\varphi} \cdot \frac{\partial f}{\partial w};$$

but by Lemma 1 as we know $U\dfrac{\lambda}{\varphi}=0$, $U\dfrac{\mu}{\varphi}=0$, and $U\dfrac{\nu}{\varphi}=\alpha w+\beta$, we have

$$Uf\equiv(\alpha w+\beta)\frac{\partial f}{\partial w},$$

or simply
$$Uf\equiv\frac{\partial f}{\partial \tau}$$

by putting $\tau=\dfrac{1}{\alpha}\log(\alpha w+\beta)$ or $\tau=\dfrac{W}{\beta}$ according as $\alpha\neq 0$ or $\alpha=0$.

Then the equations of motion become by the above change of variables

$$\frac{d^2u}{d\tau^2}=F_1\left(u,v,\tau,\frac{du}{d\tau},\frac{dv}{d\tau}\right),$$

$$\frac{d^2v}{d\tau^2}=F_2\left(u,v,\tau,\frac{du}{d\tau},\frac{dv}{d\tau}\right);$$

but since they admit of the infinitesimal transformation $Uf\equiv\dfrac{\partial f}{\partial\tau}$, F_1 and F_2 must be the functions of u, v, $\dfrac{du}{d\tau}$ and $\dfrac{dv}{d\tau}$ only, so that we have for the equations of motion

$$\left.\begin{aligned}\frac{d^2u}{d\tau^2}&=F_1\left(u,v,\frac{du}{d\tau},\frac{dv}{d\tau}\right)\\ \frac{d^2v}{d\tau^2}&=F_2\left(u,v,\frac{du}{d\tau},\frac{dv}{d\tau}\right)\end{aligned}\right\},\tag{13}$$

where either F_1 or F_2 is arbitrary; and also we know from (12) that the first integral of (13) must be

$$u\,dv-v\,du=(\alpha_1u+\beta_1v)\,d\tau.$$

Next, eliminating τ from (13),

$$\frac{d^2v}{du^2}=\frac{\dfrac{d^2v}{d\tau^2}\dfrac{du}{d\tau}-\dfrac{d^2u}{d\tau^2}\dfrac{dv}{d\tau}}{\left(\dfrac{du}{d\tau}\right)^3}=\frac{\dfrac{du}{d\tau}\cdot F_2-\dfrac{dv}{d\tau}\cdot F_1}{\left(\dfrac{du}{d\tau}\right)^3}=\left(F_2-F_1\frac{dv}{du}\right)\left(\frac{d\tau}{du}\right)^2;$$

but by (12), we have

$$\frac{d^2 v}{du^2} = \left(F_2 - F_1 \frac{dv}{du}\right) \frac{\left(u \dfrac{dv}{du} - v\right)^2}{(\alpha_1 u + \beta_1 v)^2}. \tag{14}$$

But, though F_1 (or F_2) is an arbitrary function of $u, v, \dfrac{du}{d\tau}, \dfrac{dv}{d\tau}, t$ can be considered to be the function of $u, v, \dfrac{dv}{du}$ only, since from (12), it follows

$$\frac{du}{d\tau} = \frac{\alpha_1 u + \beta_1 v}{u \dfrac{dv}{du} - v}, \quad \frac{dv}{d\tau} = \frac{\alpha_1 u + \beta_1 v}{u \dfrac{dv}{du} - v} \cdot \frac{dv}{du}.$$

Hence (14) may be written

$$\frac{d^2 v}{du^2} = \phi\left(u, v, \frac{dv}{du}\right),$$

where ϕ is considered to be any function of the arguments.

Now if we use the term "the motion admits Uf" when the equations of motion admit of an infinitesimal transformation Uf, then we have

Theorem 6: If the most general central motion admits of an infinitesimal transformation

$$Uf \equiv \left(\varphi_y + \frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x} + \left(-\varphi_x + \frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial t},$$

then the equations of motion are reduced to

$$\frac{d^2 v}{du^2} = \phi\left(u, v, \frac{dv}{du}\right)$$

and

$$udv - vdu = (\alpha_1 u + \beta_1 v)\, d\tau,$$

by putting

$$u = \frac{\lambda}{\varphi}, \quad v = \frac{\mu}{\varphi}, \quad \tau = \frac{1}{\alpha} \log\left(\alpha \frac{\nu}{\varphi} + \beta\right)$$

$$\text{or } \tau = \frac{1}{\beta} \cdot \frac{\nu}{\varphi} \text{ according as } \alpha \neq 0 \text{ or } \alpha = 0,$$

where λ, μ, ν are the functions of x, y, t, satisfying the following six relations

$$[\lambda, \varphi]=0,\ [\mu, \varphi]=0,\ [\nu, \varphi]=\alpha\nu+\beta\varphi,$$
$$[\mu, \nu]=0,\ [\nu, \lambda]=0,\ [\lambda, \mu]=\alpha_1 \lambda+\beta_1\mu.$$

From the above results we may easily obtain the solution of the equation $AF=0$; for, from

$$\frac{d^2 v}{du^2}=\phi\left(u, v, \frac{dv}{du}\right),$$

we have

$$\dot{u}^3 . \frac{d^2 v}{du^2}=F . \{\dot{u}(xv_x+yv_y)-\dot{v}(xu_x+yu_y)\}+\dot{u}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 v$$
$$-\dot{v}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 u=\dot{u}^3 . \phi\left(u, v, \frac{\dot{v}}{\dot{u}}\right),$$

hence

$$F . \{k(u_y v_x-u_x v_y)-x(v_t u_x-v_x u_t)-y(v_t w_y-v_y w_t)\}$$
$$=\dot{v}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 u-\dot{u}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 v$$
$$+\dot{u}^3\phi\left(u, v, \frac{\dot{v}}{\dot{u}}\right);$$

that is,

$$-F . \frac{k}{\varphi}(\alpha_1 u+\beta_1 v)=\dot{v}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 u$$
$$-\dot{u}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 v+\dot{u}^3\phi,$$

hence

$$F=\left\{\dot{u}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 v-\dot{v}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 u\right.$$
$$\left.-\dot{u}^3 . \phi\right\}\Big/\frac{k}{\varphi}(\alpha_1 u+\beta_1 v).$$

So we have

Theorem 7: If the equations of motion

$$\ddot{x}=x\,.\,F(x, y, t, \dot{x}, \dot{y}),\ \ \ddot{y}=y\,.\,F(x, y, t, \dot{x}, \dot{y})$$

admit of Uf, then we must have

$$F\equiv\left\{\dot{u}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 v-\dot{v}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 u\right.$$

$$\left.-\dot{u}^3\,.\,\phi\left(u, v, \frac{du}{d\tau}\cdot\frac{dv}{d\tau}\right)\right\}\Big/\frac{k}{\varphi}(\alpha_1 u+\beta_1 v).$$

Next we may consider the case, where the motion admits of two infinitesimal transformations

$$U_1 f\equiv\left(\varphi_y+\frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x}+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

$$U_2 f\equiv\left(\psi_y+\frac{x}{k}\psi_t\right)\frac{\partial f}{\partial x}+\left(-\psi_x+\frac{y}{k}\psi_t\right)\frac{\partial f}{\partial y}+\bar{\zeta}\frac{\partial f}{\partial t}.$$

By Theorem 5, there exists a relation between φ and ψ such as

$$[\psi, \varphi]=\alpha\psi+\beta\varphi,$$

hence three cases are to be considered:

1. $\alpha\neq 0,\qquad \beta\neq 0,$
2. $\alpha=0,\qquad \beta\neq 0,$
3. $\alpha=0,\qquad \beta=0.$

1. $[\psi, \varphi]=\alpha\psi+\beta\varphi,$ *i. e.* $(U_1, U_2)\equiv\alpha\,.\,U_2+\beta\,.\,U_1.$

By Lemma 3, we can find the functions λ, μ, so that

$$[\lambda, \varphi]=0,\ [\lambda, \psi]=0,\ [\lambda, \mu]=\alpha_1\lambda+\beta_1\mu,$$
$$[\mu, \varphi]=0,\ [\mu, \psi]=0;$$

and if we change the variables into $U_1 f$ and $U_2 f$ such as

$$U=\frac{\lambda}{\varphi},\ v=\frac{\mu}{\varphi},\ \tau=\frac{1}{\alpha}\log\left(\alpha\frac{\psi}{\varphi}+\beta\right),$$

then we have

$$U_1 f\equiv\frac{\partial f}{\partial\tau}\ \text{ and }\ U_2 f\equiv u e^{\alpha\tau}\frac{\partial f}{\partial u}+v e^{\alpha\tau}\frac{\partial f}{\partial v}+\frac{1}{\alpha}\left(e^{\alpha\tau}-\beta\right)\frac{\partial f}{\partial\tau}.$$

Again changing the variables as

$$\xi=\log u,\ \ \eta=\log v,\ \ \tau=\tau,$$

we have

$$U_1 f\equiv\frac{\partial f}{\partial\tau},\ \ U_2 f\equiv e^{\alpha\tau}\frac{\partial f}{\partial\xi}+e^{\alpha\tau}\frac{\partial f}{\partial\eta}+\frac{1}{\alpha}\left(e^{\alpha\tau}-\beta\right)\frac{\partial f}{\partial\tau},$$

and the equations of motion become

$$\frac{d^2\xi}{d\tau^2}=F_1\left(\xi,\eta,\frac{d\xi}{d\tau},\frac{d\eta}{d\tau}\right),$$

$$\frac{d^2\eta}{d\tau^2}=F_2\left(\xi,\eta,\frac{d\xi}{d\tau},\frac{d\eta}{d\tau}\right); \tag{15}$$

and the first integral becomes

$$\frac{d\eta}{d\tau}-\frac{d\xi}{d\tau}=\alpha_1 e^{-\eta}-\beta_1 e^{-\xi}. \tag{16}$$

But since (15) admits of $U_2 f$, we have the condition in the similar way as we have had (a); that is,

$$\frac{\partial F_1}{\partial\xi}+\frac{\partial F_1}{\partial\eta}-(\xi'-\alpha)\frac{\partial F_1}{\partial\xi'}-(\eta'-\alpha)\frac{\partial F_1}{\partial\eta'}=-2F_1-\alpha(\xi'-\alpha),$$

$$\frac{\partial F_2}{\partial\xi}+\frac{\partial F_2}{\partial\eta}-(\xi'-\alpha)\frac{\partial F_2}{\partial\xi'}-(\eta'-\alpha)\frac{\partial F_2}{\partial\eta'}=-2F_2-\alpha(\eta'-\alpha),$$

where

$$\xi'\equiv\frac{d\xi}{d\tau},\ \ \eta'\equiv\frac{d\eta}{d\tau}.$$

From the first of the above two we have the four independent solutions:

$$X=\eta-\xi,\ \ Y=\frac{\eta'-\alpha}{\xi'-\alpha},\ \ Z=e^{\xi}(\xi'-\alpha),\ \ F_1=-\alpha(\xi'-\alpha)+c(\xi'-\alpha)^2,$$

but there exists a relation between first three X, Y, Z, if we are allowed to use (16); for,

$$Y-1=\frac{\eta'-\xi'}{\xi'-\alpha}=\frac{\alpha_1 e^{-(\eta-\xi)}+\beta_1}{(\xi'-\alpha)e^{\xi}},\ \text{ that is, }\ Y-1=\frac{\alpha_1 e^{X}+\beta_1}{Z}.$$

The same thing may be said for the solution of F_2, so the equations (15) are to be written as follows:

$$\xi''=-\alpha(\xi'-\alpha)+(\xi'-\alpha)^2\,\Phi_1\left(\eta-\xi,\ \frac{\eta'-\alpha}{\xi'-\alpha}\right),$$

$$\eta''=-\alpha(\eta'-\alpha)+(\eta'-\alpha)^2\,\Phi_2\left(\eta-\xi,\ \frac{\eta'-\alpha}{\xi'-\alpha}\right),$$

whose first integral is $\eta'-\xi'=\alpha_1 e^{-\eta}+\beta_1 e^{-\xi}$; where only one of the two functions Φ_1 and Φ_2 is arbitrary.

Again putting $\bar{\xi}=\xi-\alpha\tau$, $\bar{\eta}=\eta-\alpha\tau$ and eliminating τ we have

$$\frac{d^2\bar{\eta}}{d\bar{\xi}^2}=\left[\Phi_2\left(\bar{\eta}-\bar{\xi},\ \frac{d\bar{\eta}}{d\bar{\xi}}\right)-\frac{d\bar{\eta}}{d\bar{\xi}}\cdot\Phi_1\left(\bar{\eta}-\bar{\xi},\ \frac{d\bar{\eta}}{d\bar{\xi}}\right)\right]\cdot\frac{d\bar{\eta}}{d\bar{\xi}},$$

or

$\dfrac{d^2\bar{\bar{\eta}}}{d\bar{\xi}^2}=\Phi\left(\bar{\bar{\eta}},\ \dfrac{d\bar{\bar{\eta}}}{d\bar{\xi}}\right)$; where $\bar{\bar{\eta}}\equiv\bar{\eta}-\bar{\xi}$ and Φ is any function. So we have

Theorem 8: If the most general central motion admits of two infinitesimal transformations

$$U_1f\equiv\left(\varphi_y+\frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x}+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

$$U_2f\equiv\left(\psi_y+\frac{x}{k}\psi_t\right)\frac{\partial f}{\partial x}+\left(-\psi_x+\frac{y}{k}\psi_t\right)\frac{\partial f}{\partial y}+\bar{\zeta}\frac{\partial f}{\partial t},$$

having a relation $(U_1, U_2)\equiv\alpha U_2+\beta U_1$, then the equations of motion are reduced to

$$\frac{d^2Y}{dX^2}=\Phi\left(Y,\frac{dY}{dX}\right),\quad udv-vdu=(\alpha_1u+\beta_1v)\,d\tau,$$

where

$$X\equiv\log u-\alpha\tau,\quad Y\equiv\log\frac{v}{u}.$$

2. The case, where $[\psi,\varphi]=\beta\varphi$, that is, $(U_1, U_2)\equiv\beta U_1$.

This case is easily reducible to the former, for if we interchange φ and ψ, and put $\beta=0$ in the case 1, then we have our case. So we have

Theorem 9: If the most general central motion admits of two infinitesimal transformations, having a relation

$(U_1, U_2) \equiv \beta U_1$, then the equations of motion are reduced to

$$\frac{d^2Y}{dX^2} = \Phi\left(Y, \frac{dY}{dX}\right), \ udv - vdu = (\alpha_1 u + \beta_1 v)\, d\tau$$

by the substitutions

$$u = \frac{\lambda}{\psi}, \ v = \frac{\mu}{\psi}, \ \tau = \frac{1}{\beta} \cdot \frac{\varphi}{\psi},$$

where

$$X \equiv \log u - \log \tau, \ \ Y \equiv \log \frac{v}{u}.$$

3. The case, where $[\psi, \varphi] = 0$, *i. e.* $(U_1, U_2) \equiv 0$.

By Lemma 3, we can find two functions μ and ν, which satisfy

$$[\mu, \varphi] = 0, \ [\nu, \varphi] = \alpha\nu + \beta\varphi, \ [\mu, \nu] = 0,$$

$$[\mu, \psi] = -(\alpha_1 \psi + \beta_1 \varphi), \ [\nu, \psi] = 0.$$

Therefore, if we put $u = \dfrac{\psi}{\varphi}, \ v = \dfrac{\mu}{\varphi}, \ \tau = \dfrac{1}{2} \log\left(\alpha \dfrac{\psi}{\varphi} + \beta\right)$,

then

$$U_1 f \equiv \frac{\partial f}{\partial \tau}, \ U_2 f \equiv (\alpha_1 u + \beta_1 v) \frac{\partial f}{\partial v};$$

and the equations of motion must take the form, from the invariancy by $U_1 f$, such that

$$\frac{d^2 u}{d\tau^2} = F_1\left(u, v, \frac{du}{d\tau}, \frac{dv}{d\tau}\right), \ \frac{d^2 v}{d\tau^2} = F_2\left(u, v, \frac{du}{d\tau}, \frac{dv}{d\tau}\right), \qquad (17)$$

having $u\,dv - v\,du = (\alpha_1 u + \beta_1 v)\, d\tau$ (18) as the first integral.

But as they again admit of $U_2 f$, we have

$$(\alpha_1 u + \beta_1 v) \frac{\partial F_1}{\partial v} + (\alpha_1 u' + \beta_1 v') \frac{\partial F_1}{\partial v'} = 0,$$

$$(\alpha_1 u + \beta_1 v) \frac{\partial F_2}{\partial v} + (\alpha_1 u' + \beta_1 v') \frac{\partial F_2}{\partial v'} = \alpha_1 \cdot F_1 + \beta_1 \cdot F_2,$$

where $u' \equiv \dfrac{du}{d\tau}, \ v' \equiv \dfrac{dv}{d\tau}$. Solving the above, we have

$$F_1 = u'^2 \cdot \Phi_1(u, u', \eta'),\ \ F_2 \equiv \frac{1}{\beta_1}\{u'^2 \cdot \Phi_2(u, u', \eta') + \eta'^2\} - \frac{\alpha_1}{\beta_1} \cdot u'^2 \cdot \Phi_1,$$

where $$\eta = \log(\alpha_1 u + \beta_1 v).$$

But using (18) we see that u, u' and η' are connected by a relation ; for,

$$\eta' - \frac{u'}{u} = \frac{\beta_1}{u} \cdot \frac{uv' - vu'}{\alpha_1 u + \beta_1 v} = \frac{\beta_1}{u},$$

hence $u\eta' - u' = \beta_1$ (19). So the equations of motion become

$$u'' = \frac{\alpha_1}{\beta_1} u'^2 \cdot \Phi_1\left(u, \frac{\eta'}{u'}\right),$$

$$\eta'' = \frac{1}{\beta_1}\left\{u'^2 \cdot \Phi_2\left(u, \frac{\eta'}{u'}\right) + \eta'^2\right\} - \frac{\alpha_1}{\beta_1} u'^2\, \Phi_1\left(u, \frac{\eta'}{u'}\right),$$

and, from (19) we have the first integral

$$u\eta' - u' = \beta_1.$$

But we see easily that the above two equations may be replaced by the following :

$$u'' = \frac{\alpha_1}{\beta_1} u'^2 \cdot \Phi_1\left(u, \frac{d\eta}{du}\right),\ \eta'' = u'^2 \cdot \Phi_3\left(u, \frac{d\eta}{du}\right);$$

therefore, eliminating τ from them, we have

$$\frac{d^2\eta}{du^2} = \frac{\alpha_1}{\beta_1} \cdot \frac{d\eta}{du} \cdot \Phi_1\left(u, \frac{d\eta}{du}\right) - \Phi_3\left(u, \frac{d\eta}{du}\right),$$

that is,

$$\frac{d^2\eta}{du^2} = \Phi\left(u, \frac{d\eta}{du}\right),$$

which is soluble by the integration of a linear differential equation and a quadrature.

So we have

Theorem 10: If the most general central motion admits of two infinitesimal transformations $U_1 f$, $U_2 f$ having a relation $(U_1, U_2) \equiv 0$, then the equations of motion are reduced to

$$\frac{d^2\eta}{du^2}=\Phi\left(u, \frac{d\eta}{du}\right), \ u d\eta - du = \beta_1 d\tau,$$

by a substitution

$$u=\frac{\psi}{\varphi}, \ \eta=\log\frac{\alpha_1\psi+\beta_1\mu}{\varphi}, \ \tau=\frac{1}{2}\log\left(\alpha\frac{\nu}{\varphi}+\beta\right),$$

where μ and ν satisfy the following five relations:

$$[\mu, \varphi]=0, \ [\nu, \varphi]=\alpha\nu+\beta\varphi, \ [\mu, \nu]=0,$$
$$[\psi, \mu]=\alpha_1\psi+\beta_1\mu, \ [\nu, \psi]=0.$$

Remarks: In the equations of motion

$$\ddot{x}=x\,.\,F(x, y, t, \dot{x}, \dot{y}), \ \ddot{y}=y\,.\,F(x, y, t, \dot{x}, \dot{y}),$$

which admit of two infinitesimal transformations, we can easily find the form of the function F, *i.e.* the common solution of $AF=0$ and $BF=0$.

In the case 1 and 2, since th equation of motion is

$$\frac{d^2\bar{\bar{\eta}}}{d\bar{\xi}^2}=\Phi\left(\bar{\bar{\eta}}, \frac{d\bar{\bar{\eta}}}{d\bar{\xi}}\right),$$

we have

$$\begin{aligned}\dot{\bar{\xi}}^3\,.\,\Phi\left(\bar{\bar{\eta}}, \frac{d\bar{\bar{\eta}}}{d\bar{\xi}}\right) &= \ddot{\bar{\bar{\eta}}}\dot{\bar{\xi}}-\ddot{\bar{\xi}}\dot{\bar{\bar{\eta}}} \\ &= \{\dot{\bar{\xi}}(x\bar{\bar{\eta}}_x+y\bar{\bar{\eta}}_y)-\dot{\bar{\bar{\eta}}}(x\bar{\xi}_x+y\bar{\xi}_y)\}\,F \\ &\quad +\dot{\bar{\xi}}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2\bar{\bar{\eta}} \\ &\quad -\dot{\bar{\bar{\eta}}}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2\bar{\xi}\,;\end{aligned}$$

but in an analogous manner as we used in obtaining Theorem 7, we have

$$\begin{aligned}&\dot{\bar{\xi}}(x\bar{\bar{\eta}}_x+y\bar{\bar{\eta}}_y)-\dot{\bar{\bar{\eta}}}(x\bar{\xi}_x+y\bar{\xi}_y) \\ &\qquad = k(\bar{\xi}_y\bar{\bar{\eta}}_x-\bar{\xi}_x\bar{\bar{\eta}}_y)-x(\bar{\xi}_t\bar{\bar{\eta}}_x-\bar{\xi}_x\bar{\bar{\eta}}_t)-y(\bar{\xi}_t\bar{\bar{\eta}}_y-\bar{\xi}_y\bar{\bar{\eta}}_t),\end{aligned}$$

and substituting the values of $\bar{\xi}$ and $\bar{\bar{\eta}}$, *i.e.* $\bar{\xi}=\log u-\alpha\tau$, $\bar{\bar{\eta}}=\log\dfrac{v}{u}$,

$$=-\frac{k}{\varphi uv}(\alpha_1 u+\beta_1 v).$$

Hence we have the required form of F:

$$F=\left\{\dot{\bar{\bar{\eta}}}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2\bar{\bar{\xi}}-\dot{\bar{\bar{\xi}}}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2\bar{\bar{\eta}}\right.$$

$$\left.+\dot{\bar{\bar{\xi}}}\,.\,W\left(\bar{\bar{\eta}},\frac{d\bar{\bar{\eta}}}{d\bar{\bar{\xi}}},\ x\dot{y}-y\dot{x}\right)\right\}\Big/-\frac{k(\alpha_1 u+\beta_1 v)}{\varphi uv},$$

assuming that $x\dot{y}-y\dot{x}=k$.

In the case 3, similarly to the above, we have the form of F:

$$F=\left\{\dot{\eta}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2 u-\dot{u}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^2\eta\right.$$

$$\left.+\dot{u}^3\,.\,W\left(u,\frac{d\eta}{d\tau},\frac{du}{d\tau}\right)\right\}\Big/-\frac{k\beta_1}{\varphi}e^{\eta}.$$

III. The infinitesimal transformations and the integration of the equations of motion, when F does not contain t.

In the central motions, the most ordinary case which occurs is that where the equations of motion take the form

$$\ddot{x}=x\,.\,F(x,y,\dot{x},\dot{y}),\quad \ddot{y}=y\,.\,F(x,y,\dot{x},\dot{y}).$$

If the infinitesimal transformation, which the equations of motion admit, is

$$Uf=\left(\varphi_y+\frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial x}+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

then by Theorem 3, F is a solution of the following equation:

$$(k\varphi_y+x\varphi_t)\frac{\partial F}{\partial x}+(-k\varphi_x+y\varphi_t)\frac{\partial F}{\partial y}+k\zeta\frac{\partial F}{\partial t}+(k\dot{\varphi}_y+\dot{x}\varphi_t+x\dot{\varphi}_t-k\dot{x}\dot{\zeta})\frac{\partial F}{\partial \dot{x}}$$

$$+(-k\dot{\varphi}_x+\dot{y}\varphi_t+y\dot{\varphi}_t-k\dot{y}\dot{\zeta})\frac{\partial F}{\partial \dot{y}}+(3k\dot{\zeta}-2\varphi_t)F$$

$$=\left(\dot{x}\frac{\partial}{\partial y}+\dot{y}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^3\varphi. \tag{20}$$

But by the hypothesis that F does not contain t, we have by differentiating (20) i-times with respect to t,

$$(k\varphi_{ty}^i + x\varphi_t^{i+1})\frac{\partial F}{\partial x} + (-k\varphi_{tx}^i + y\varphi_t^{i+1})\frac{\partial F}{\partial y} + k\zeta_t^i\frac{\partial F}{\partial t}$$

$$+(k\dot{\varphi}_{ty}^i + \dot{x}\varphi_t^{i+1} + x\dot{\varphi}_t^{i+1} - k\dot{x}\zeta_t^i)\frac{\partial F}{\partial \dot{x}} + (-k\dot{\varphi}_{tx}^i + \dot{y}\varphi_t^{i+1} + y\dot{\varphi}_t^{i+1} - k\dot{y}\zeta_t^i)$$

$$+(3\,k\dot{\zeta}_t^i - 2\,\varphi_t^{i+1})\,F = \left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)^3 \varphi_t^i, \qquad (21)$$

$$i = 0, 1, 2, \cdots\cdots.$$

We know, therefore, that F is the common solution of (21), when we change all values of i, so that we have by Theorem 5

$$[\varphi_t^i, \varphi_t^j] = \alpha_{ij}(\varphi_t^i - \varphi_t^j), \quad \begin{array}{l} i=0, 1, 2, \cdots\cdots, \\ j=0, 0, 2, \cdots\cdots, \end{array} \qquad (22)$$

where α_{ij} is a constant.

First we shall show $\alpha_{ij} = 0$. For, if we assume $\alpha_{ij} \neq 0$, then taking specially $i=0$ and $j=1$, we have

$$[\varphi, \varphi_t] = \alpha_{01}(\varphi - \varphi_t);$$

differentiating with respect to t

$$[\varphi, \varphi_{tt}] = \alpha_{01}(\varphi_t - \varphi_{tt}),$$

therefore

$$\alpha_{02}(\varphi - \varphi_{tt}) = \alpha_{01}(\varphi_t - \varphi_{tt}).$$

Hence we can find φ from the above, which is of the form $A\,e^{\alpha t}$, where A is a function of x, y, and α is a constant.

Substituting the value of φ into (18), we see that $\alpha_{ij} = 0$, which is contradictory. So the assumption is proved.

Hence $[\varphi_t^i, \varphi_t^j] = 0$, and if we take $i=0$, $j=1, 2, 3, \cdots\cdots$, then using Lemma 1, we have

$$\left(\varphi_y + \frac{x}{k}\varphi_t\right)\frac{\partial}{\partial x}\left(\frac{\varphi_t^j}{\varphi}\right) + \left(-\varphi_x + \frac{y}{k}\varphi_t\right)\frac{\partial}{\partial y}\left(\frac{\varphi_t^j}{\varphi}\right)$$

$$+\zeta\frac{\partial}{\partial t}\left(\frac{\varphi_t^j}{\varphi}\right) = 0 \quad j=1, 2, 3, \cdots\cdots;$$

hence it must be

$$\frac{\partial}{\partial x}\left(\frac{\varphi_t^j}{\varphi}\right)=\frac{\partial}{\partial y}\left(\frac{\varphi_t^j}{\varphi}\right)=\frac{\partial}{\partial t}\left(\frac{\varphi_t^j}{\varphi}\right)=0.$$

We have, therefore, $\varphi_t^j=\lambda\varphi\ (j=1, 2,\cdots\cdots)$, where λ is any constant; hence $\varphi=Ae^{\lambda t}$, where A is any function of x, y.

So we have

Theorem 11: If in the equations of motion

$$\ddot{x}=x\,.\,F,\ \ddot{y}=y\,.\,F,$$

F is the function of $x, y,\ \dot{x}, \dot{y}$ and they admit of an infinitesimal transformation

$$Uf\equiv\left(\varphi_y+\frac{x}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\left(-\varphi_x+\frac{y}{k}\varphi_t\right)\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

then it is necessary and sufficient that

$$\varphi=Ae^{\lambda t},$$

where A is any function of x, y, and λ is any constant.

But in this case, I have not yet any successful result for the integration of the equations of motion, so we consider the next special important case.

IV. The special case when $\lambda=0$.

In Theorem 11, if $\lambda=0$, then φ is any function of x, y and the infinitesimal transformation is

$$Uf\equiv\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

where $\zeta\equiv\frac{1}{k}(2\varphi-x\varphi_x-y\varphi_y)$. If the equations of motion are

$$\left.\begin{aligned}\ddot{x}&=x\,.\,F(x, y, \dot{x}, \dot{y})\\ \ddot{y}&=y\,.\,F(x, y, \dot{x}, \dot{y})\end{aligned}\right\}, \tag{23}$$

then F must satisfy $AF=0$; that is,

$$\varphi_y\frac{\partial F}{\partial x}-\varphi_x\frac{\partial F}{\partial y}+(\dot{\varphi}_y-\dot{x}\dot{\zeta})\frac{\partial F}{\partial \dot{x}}+(-\dot{\varphi}_x-\dot{y}\dot{\zeta})\frac{\partial F}{\partial \dot{y}}+3\,\dot{\zeta}F$$

$$=\frac{1}{k}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^3 F. \tag{24}$$

Then we shall consider some properties of our infinitesimal transformation and next, using them, the integrations of the equations of motion.

First, we show the next

Theorem 12: If F is a solution of the equation

$$\varphi_y \frac{\partial F}{\partial x} - \varphi_x \frac{\partial F}{\partial y} + (\dot{\varphi}_y - \dot{x}\zeta)\frac{\partial F}{\partial \dot{x}} + (-\dot{\varphi}_x - \dot{y}\zeta)\frac{\partial F}{\partial \dot{y}} + 3\zeta F$$
$$= \frac{1}{k}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^3 F,$$

then the system of curves defined by the equations of motion

$$\ddot{x} = x \,.\, F(x, y, \dot{x}, \dot{y}),\ \ \ddot{y} = y \,.\, F(x, y, \dot{x}, \dot{y})$$

admits of the infinitesimal point-transformation

$$\overline{U}f \equiv \varphi_y \frac{\partial f}{\partial x} - \varphi_x \frac{\partial f}{\partial y}.$$

To prove this, we must use the Lie's theorem(1):

Die Differentialgleichung zweiter Ordnung $\ddot{y} = \omega(x, y, \dot{y})$ gestattet dann und nur dann die eingliedrige Gruppe $\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$, wenn der Ausdruck

$$(\eta_y - 2\xi_x - 3\xi_y y')\omega - \xi_{yy} y'^3 + (\eta_{yy} - 2\xi_{xy})y'^2 + (2\eta_{xy} - \xi_{xx})y' + \eta_{xx}$$
$$-\xi\frac{\partial \omega}{\partial x} - \eta\frac{\partial \omega}{\partial y} - (\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2)\frac{\partial \omega}{\partial y'} \equiv 0 \qquad (25)$$

ist für alle Werte von x, y, y'.

From $$x\dot{y} - y\dot{x} = k \text{ and } y' = \frac{\dot{y}}{\dot{x}},$$

we have

$$\dot{x} = \frac{k}{xy' - y},\ \ \dot{y} = \frac{ky'}{xy' - y}; \qquad (26)$$

and therefore eliminating t from (23),

$$y'' = -\frac{1}{k^2}(xy' - y)^3 \,.\, \Phi(x, y, y'), \qquad (27)$$

(1) S. Lie: Vorlesungen über Differentialgleichungen, S. 363.

where $\Phi \equiv F\left(x, y, \frac{k}{xy'-y}, \frac{ky'}{xy'-y}\right)$. The equation (27) defines the curves of our motion.

Now applying the Lie's theorem, in our case, *i. e.* $\omega \equiv \frac{1}{k^2}(xy'-y)^3 . \Phi$, $\xi = \varphi_y$, and $\eta = -\varphi_x$, the left side of (25) becomes

$$\varphi_y \frac{\partial \omega}{\partial x} - \varphi_x \frac{\partial \omega}{\partial y} - \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y}\right)^2 \varphi \cdot \frac{\partial \omega}{\partial y'} + 3(\varphi_{xy} + y'\varphi_{yy})\,\omega$$
$$+ \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y}\right)^3 \varphi. \tag{28}$$

But from (26), we have

$$-k^2 \frac{\partial \omega}{\partial x} = 3(xy'-y)^2 \cdot y' \cdot F + (xy'-y)^3 \left(\frac{\partial F}{\partial x} - \frac{\dot{x}\dot{y}}{k} \frac{\partial F}{\partial \dot{x}} - \frac{\dot{y}^2}{k} \frac{\partial F}{\partial \dot{y}}\right),$$

$$-k^2 \frac{\partial \omega}{\partial y} = -3(xy'-y)^2 \cdot F + (xy'-y)^3 \left(\frac{\partial F}{\partial y} + \frac{\dot{x}^2}{k} \frac{\partial F}{\partial \dot{x}} + \frac{\dot{x}\dot{y}}{k} \frac{\partial F}{\partial \dot{y}}\right),$$

$$-k^2 \frac{\partial \omega}{\partial y'} = 3(xy'-y)^2 \cdot x \cdot F + (xy'-y)^3 \left\{-\frac{\dot{x}^2}{k}\left(x \frac{\partial F}{\partial \dot{x}} + y \frac{\partial F}{\partial \dot{y}}\right)\right\};$$

hence (28) becomes

$$-\frac{1}{k^2}(xy'-y)^3 \left\{\varphi_y \frac{\partial F}{\partial x} - \varphi_x \frac{\partial F}{\partial y} + (\dot{\varphi}_y - \dot{x}\dot{\zeta}) \frac{\partial F}{\partial \dot{x}} + (-\dot{\varphi}_x - \dot{y}\dot{\zeta}) \frac{\partial F}{\partial \dot{y}}\right.$$
$$\left. + 3\dot{\zeta} F - \frac{1}{k}\left(\dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y}\right)^3 \varphi\right\},$$

which vanishes identically by the hypothesis. Thus the theorem is proved.

It is to be noticed that $\varphi_y \frac{\partial f}{\partial x} - \varphi_x \frac{\partial f}{\partial y}$ is the area-invariant infinitesimal transformation and the paths of the transformation are $\varphi = \text{const.}$.

Next, by $Uf \equiv \varphi_y \frac{\partial f}{\partial x} - \varphi_x \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial t}$ it is meant that

$$\delta x = \varphi_y\, \delta a, \quad \delta y = -\varphi_x\, \delta a, \quad \delta t = \zeta\, \delta a, \tag{29}$$

where $\delta x, \delta y, \delta t$ are the variations of x, y, t for the variation of the parameter a respectively; hence eliminating the parameter from (29), we have

$$\frac{\delta x}{\delta t}=\frac{\varphi_y}{\zeta},\ \frac{\delta y}{\delta t}=-\frac{\varphi_x}{\zeta}.$$

We know, therefore, that the system of the central curves is transformed into itself by a motion

$$\dot{x}=\frac{\varphi_y}{\zeta},\ \dot{y}=-\frac{\varphi_x}{\zeta} \tag{30}$$

of the points on the central curves.

Next we consider the case when the above motion is central. From (30), it follows

$$x\dot{y}-y\dot{x}=-\frac{x\varphi_x+y\varphi_y}{\zeta}$$
$$=\frac{k\zeta-2\,\varphi}{\zeta},$$

but the above expression must become constant when φ equals to a constant; hence ζ must be a function of φ only. So we have

Theorem 13: If ζ is a function of φ only, the system of central curves having the relation $x\dot{y}-y\dot{x}=k$ is transformed into itself by another central motion expressed by

$$\dot{x}=\frac{\varphi_y}{\zeta},\ \dot{y}=-\frac{\varphi_x}{\zeta}.$$

Next we consider the case in which a system of the orthogonal trajectories of $\varphi=$const. may be invariant by

$$Uf\equiv\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t}.$$

The orthogonal trajectories of $\varphi=$const. are expressed by $\varphi_y\dot{x}-\varphi_x\dot{y}=0$, hence we have as the condition of invariancy by Uf

$$(\varphi_{xy}\varphi_y-\varphi_{yy}\varphi_x)\dot{x}-(\varphi_{xx}\varphi_y-\varphi_{xy}\varphi_x)\dot{y}+\varphi_y(\dot{\varphi}_y-\dot{x}\dot{\zeta})-\varphi_x(-\dot{\varphi}_x-\dot{y}\dot{\zeta})=0.$$

Substituting the value of $\dot{x}:\dot{y}$ into the above,

$$(\varphi_x^2-\varphi_y^2)(\varphi_{xx}-\varphi_{yy})+4\,\varphi_x\varphi_y\varphi_{xy}=0 \tag{31}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\varphi_x}{\varphi_x^2+\varphi_y^2}\right)+\frac{\partial}{\partial y}\left(\frac{\varphi_y}{\varphi_x^2+\varphi_y^2}\right)=0;$$

hence we can find a function ψ, such that

$$\frac{\partial\psi}{\partial x}=\frac{\varphi_y}{\varphi_x^2+\varphi_y^2},\ \frac{\partial\psi}{\partial y}=-\frac{\varphi_x}{\varphi_x^2+\varphi_y^2}.$$

And also we can see that

$$(\psi_x^2-\psi_y^2)(\psi_{xx}-\psi_{yy})+4\,\psi_x\psi_y\psi_{xy}=0,$$

which shows that the orthogonal trajectories of $\psi=$const. admit of the transformation $\psi_y\frac{\partial f}{\partial x}-\psi_x\frac{\partial f}{\partial y}+\frac{1}{k}(2\,\psi-x\psi_x-y\psi_y)\frac{\partial f}{\partial t}$.

Also the solution of the partial differential equation (31) may be obtained as the envelope of

$$\varphi=xp+yq+g\,(p) \text{ and } \pm\frac{q}{p}=\tan\log\sqrt{p^2+q^2}\ ,$$

where g is an arbitrary function. So we have

Theorem 14: If φ is a solution of the differential equation

$$(\varphi_x^2-\varphi_y^2)(\varphi_{xx}-\varphi_{yy})+4\,\varphi_x\varphi_y\varphi_{xy}=0, \tag{A}$$

then the infinitesimal transformation

$$Uf\equiv\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t}$$

has an invariant system $\psi=$const. orthogonal to $\varphi=$const., where φ is another solution of (A); and the same holds good when φ and ψ are interchanged.

Next, we consider the integration of the equations of motion when they admit of an infinitesimal transformation. In our case, it is somewhat different from the general case that we stated before.

If the equations of motion are

$$\ddot{x}=x\,.\,F(x,y,\dot{x},\dot{y}),\ \ddot{y}=y\,.\,F(x,y,\dot{x},\dot{y})$$

and the infinitesimal transformation is

$$Uf\equiv\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},$$

then F is the solution of

$$AF \equiv \varphi_y \frac{\partial F}{\partial x} - \varphi_x \frac{\partial F}{\partial y} + (\dot{\varphi}_y - \dot{x}\dot{\zeta}) \frac{\partial F}{\partial \dot{x}} + (-\dot{\varphi}_x - \dot{y}\dot{\zeta}) \frac{\partial F}{\partial \dot{y}} + 3\dot{\zeta}F$$

$$-\frac{1}{k}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^3 \varphi = 0.$$

Now, we are going to solve the equation $AF=0$ directly, for the methods used in II or those like them are not applicable in this case.

For the purpose, introduce an auxiliary function $\psi(x, y)$ which satisfies the following relation

$$\varphi_x \psi_y - \varphi_y \psi_x = \lambda_1 \varphi + \lambda_2 \psi + c, \tag{32}$$

where λ_1, λ_2 and c are any constants, all of which do not vanish, then it is easily seen that

$$\overline{F} \equiv \left\{\psi\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^2 \varphi - \dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^2 \psi\right\} \Big/ -k(\lambda_1\varphi + \lambda_2\psi + c)$$

is a particular solution of $AF=0$.

For, if we substitute F into AF, then we have

$$A\overline{F} \equiv A\dot{x}^3 + 3B\dot{x}^2\dot{y} + 3C\dot{x}\dot{y}^2 + D\dot{y}^3 - \frac{1}{k}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^3 \varphi,$$

where

$$A \equiv \frac{1}{k\chi}\{\lambda_2(\psi_{xx}\varphi_x - \psi_x\varphi_{xx}) - \varphi_y\psi_x\varphi_{xxx} - \varphi_x(\psi_{xxy}\varphi_x - \psi_{xxy}\varphi_y$$
$$-\varphi_{xxy}\psi_x + 2\varphi_{xx}\psi_{xy} - 2\varphi_{xy}\psi_{xx}) + \varphi_{xx}\chi_x\},$$

$$3B \equiv \frac{1}{k\chi}\{\lambda_2(\psi_y\varphi_{xx} + 2\psi_x\varphi_{xy} - \varphi_y\psi_{xx} - 2\varphi_x\psi_{xy}) + \varphi_y\chi_{xx} + 2\varphi_x\chi_{xy}$$
$$-\chi_y\varphi_{xx} - 2\chi_x\varphi_{xy} + 3\chi\varphi_{xxy}\},$$

$$3C \equiv \frac{1}{k\chi}\{\lambda_2(\psi_x\varphi_{yy} + 2\psi_y\varphi_{xy} - \varphi_x\psi_{yy} - 2\varphi_y\psi_{xy}) + \varphi_x\psi_{yy} + 2\varphi_y\chi_{xy}$$
$$-\chi_x\varphi_{yy} - 2\chi_y\varphi_{xy} + 3\chi\varphi_{xyy}\},$$

$$D \equiv \frac{1}{k\chi}\{\lambda_2(\psi_y\varphi_{yy} - \varphi_y\psi_{yy}) + \varphi_y\chi_{yy} - \chi_y\varphi_{yy} + \chi\varphi_{yyy}\},$$

and

$$\chi \equiv \varphi_x \psi_y - \psi_x \varphi_y.$$

But from (32), we have

$$\varphi_{xxx}\psi_y + \psi_{xxy}\varphi_x - \varphi_{xxy}\psi_x - \psi_{xxx}\varphi_y + 2\,\varphi_{xx}\psi_{xy} - 2\,\varphi_{xy}\psi_{xx} = \lambda_1\,\varphi_{xx} + \lambda_2\psi_{xx},$$

therefore

$$A = \frac{1}{k\chi}\{\lambda_2(\psi_{xx}\varphi_x - \psi_x\varphi_{xx}) + (\varphi_x\psi_y - \psi_x\varphi_y)\varphi_{xxx} - \varphi_x(\lambda_1\varphi_{xx} + \lambda_2\psi_{xx})$$
$$+ \varphi_{xx}\chi_x\};$$

therefore

$$= \frac{1}{k}\varphi_{xxx}.$$

Similarly, B, C and D become $\frac{1}{k}\varphi_{xxy}$, $\frac{1}{k}\varphi_{xyy}$, $\frac{1}{k}\varphi_{yyy}$ respectively. So the propositon is proved.

Next, if we put $\overline{F} + \dot{\varphi}^3.\,\Phi$ for F into $AF=0$, then we have the equation for Φ:

$$\overline{A}\Phi \equiv \varphi_y\frac{\partial\Phi}{\partial x} - \varphi_x\frac{\partial\Phi}{\partial y} + (\varphi_y - \dot{x}\dot{\zeta})\frac{\partial\Phi}{\partial\dot{x}} + (-\varphi_x - \dot{y}\dot{\zeta})\frac{\partial\Phi}{\partial\dot{y}} = 0.$$

To solve this, change the variables as follows:

$$\varphi = \varphi(x, y),\ \ \psi = \psi(x, y),\ \ t = t, \tag{33}$$

then $\overline{A}\Phi = 0$ becomes

$$(\lambda_1\varphi + \lambda_2\psi + c)\frac{\partial\Phi}{\partial\psi} + \dot{\varphi}\dot{\zeta}\frac{\partial\Phi}{\partial\dot{\varphi}} + (\lambda_1\dot{\varphi} + \lambda_2\dot{\psi} + \dot{\psi}\dot{\zeta})\frac{\partial\Phi}{\partial\dot{\psi}} = 0;$$

the three independent solutions are easily obtained as

$$\varphi,\ \frac{d}{d\varphi}\log(\lambda_1\varphi + \lambda_2\psi + c),\ \frac{(x\varphi_x + y\varphi_y)\dot{\psi} - (x\psi_x + y\psi_y)\dot{\varphi}}{\lambda_1\varphi + \lambda_2\psi + c}$$
$$(= x\dot{y} - y\dot{x}).$$

Hence we have

$$\Phi = W\left(\varphi,\ \frac{d}{d\varphi}\log(\lambda_1\varphi + \lambda_2\psi + c),\ x\dot{y} - y\dot{x}\right),$$

so that the general solution of $AF=0$ is

$$\overline{F}+\dot{\varphi}^3 \cdot \Phi.$$

From (33) we have, by differentiating with respect to t and applying the equations of motion,

$$\ddot{\varphi}=(x\varphi_x+y\varphi_y)\cdot F+\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\varphi,$$

$$\ddot{\psi}=(x\psi_x+y\psi_y)\cdot F+\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi;$$

hence $\dfrac{d^2\psi}{d\varphi^2}=\dfrac{\ddot{\psi}\dot{\varphi}-\ddot{\varphi}\dot{\psi}}{\dot{\varphi}^3}$

$$=\left[\dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi-\dot{\psi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\varphi\right.$$

$$\left.+\{(x\psi_x+y\psi_y)\dot{\varphi}-(x\varphi_x+y\varphi_y)\dot{\psi}\}\cdot F\right]\Big/\dot{\varphi}^3.$$

Substituting the value of F into the above, we have

$$\frac{d^2\psi}{d\varphi^2}=\{k(\lambda_1\varphi+\lambda_2\psi+c)(\overline{F}+\dot{\varphi}^3\cdot W)-k(\lambda_1\varphi+\lambda_2\psi+c)\overline{F}\}\Big/\dot{\varphi}^3.$$

$$=k(\lambda_1\varphi+\lambda_2\psi+c)\cdot W.$$

Next if put $\eta\equiv\log(\lambda_1\varphi+\lambda_2\psi+c)$, then we have finally

$$\frac{d^2\eta}{d\varphi^2}=\lambda_2 W\left(\varphi,\frac{d\eta}{d\varphi},k\right)+\left(\frac{d\eta}{d\varphi}\right)^2,$$

that is,

$$\frac{d^2\eta}{d\varphi^2}=\Phi\left(\varphi,\frac{d\eta}{d\varphi}\right).$$

So we have

Theorem 15: If the equations of motion

$$\ddot{x}=x\cdot F(x,y,\dot{x},\dot{y}),\ \ \ddot{y}=y\cdot F(x,y,\dot{x},\dot{y}),$$

having the first integral $x\dot{y}-y\dot{x}=k$, admit of an infinitesimal transformation $\varphi_y\dfrac{\partial f}{\partial x}-\varphi_x\dfrac{\partial f}{\partial y}+\zeta\dfrac{\partial f}{\partial t}$, then it must be

$$F \equiv \left\{ \dot{\psi} \left(\dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} \right)^2 \varphi - \dot{\varphi} \left(\dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} \right)^2 \psi \right\} \Big/ -k(\lambda_1 \varphi + \lambda_2 \psi + c)$$

$$+ \dot{\varphi}^3 \cdot W \left\{ \varphi, \frac{d}{d\varphi} \log (\lambda_1 \varphi + \lambda_2 \psi + c) \right\},$$

where ψ is determined by the relation

$$\varphi_x \psi_y - \varphi_y \psi_x = \lambda_1 \varphi + \lambda_2 \psi + c.$$

Whence the equations are reduced to

$$\frac{d^2\eta}{d\varphi^2} = \Phi \left(\varphi, \frac{d\eta}{d\varphi} \right),$$

which is integrated by the integration of a linear differential equation and a quadrature.

Remark 1: If $\lambda_2 = 0$, then

$$\bar{A}\Phi \equiv (\lambda_1 \varphi + c) \frac{\partial \Phi}{\partial \psi} + \dot{\varphi}\dot{\zeta} \frac{\partial \Phi}{\partial \dot{\varphi}} + (\lambda_1 \dot{\varphi} + \dot{\psi}\dot{\zeta}) \frac{\partial \Phi}{\partial \dot{\psi}} = 0,$$

from which we have $\Phi \equiv W\left(\varphi, \dfrac{d\psi}{d\varphi} - \dfrac{\lambda_1 \psi}{\lambda_1 \varphi + c} \right)$; hence we have for the equations of motion

$$\frac{d^2 \psi}{d\varphi^2} = k(\lambda_1 \varphi + c) \cdot W\left(\varphi, \frac{d\psi}{d\varphi} - \frac{\lambda_1 \psi}{\lambda_1 \varphi + c} \right).$$

But if we put $\eta = \dfrac{d\psi}{d\varphi} - \dfrac{\lambda_1 \psi}{\lambda_1 \varphi + c}$, then

$$\frac{d\eta}{d\varphi} = k(\lambda_1 \varphi + c) \cdot W(\varphi, \eta) - \frac{\lambda_1}{\lambda_1 \varphi + c} \cdot \eta,$$

that is,

$$\frac{d\eta}{d\varphi} = \Phi(\varphi, \eta).$$

Remark 2: If $\lambda_1 = \lambda_2 = 0$, then

$$\bar{A}\Phi \equiv c \frac{\partial \Phi}{\partial \psi} + \dot{\varphi}\dot{\zeta} \frac{\partial \Phi}{\partial \dot{\varphi}} + \dot{\psi}\dot{\zeta} \frac{\partial \Phi}{\partial \dot{\psi}} = 0,$$

from which we have $\Phi \equiv W\left(\varphi, \dfrac{d\psi}{d\varphi} \right)$; hence we have for the equations of motion

$$\frac{d^2\psi}{d\varphi^2}=k\,.\,c\,.\,W\left(\varphi,\frac{d\psi}{d\varphi}\right).$$

Next, we may consider the case when the equation of motion admits of two infinitesimal transformations

$$U_1f\equiv\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},\quad U_2f\equiv\psi_y\frac{\partial f}{\partial x}-\psi_x\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t}.$$

Then F must be a common solution of the two partial differential equations

$$AF\equiv\varphi_y\frac{\partial F}{\partial x}-\varphi_x\frac{\partial F}{\partial y}+(\dot\varphi_y-\dot x\dot\zeta)\frac{\partial F}{\partial\dot x}+(-\dot\varphi_x-\dot y\dot\zeta)\frac{\partial F}{\partial\dot y}+3\dot\zeta F$$
$$-\frac{1}{k}\left(\dot x\frac{\partial}{\partial x}+\dot y\frac{\partial}{\partial y}\right)^3\varphi=0,$$

$$BF\equiv\psi_y\frac{\partial F}{\partial x}-\psi_x\frac{\partial F}{\partial y}+(\dot\psi_y-\dot x\dot\zeta)\frac{\partial F}{\partial\dot x}+(-\dot\psi_x-\dot y\dot\zeta)\frac{\partial F}{\partial\dot y}+3\dot\zeta F$$
$$-\frac{1}{k}\left(\dot x\frac{\partial}{\partial x}+\dot y\frac{\partial}{\partial y}\right)^3\psi=0.$$

Similarly as in the general case, if we put

$$\bar AF\equiv\varphi_y\frac{\partial F}{\partial x}-\varphi_x\frac{\partial F}{\partial y}+(\dot\varphi_y-\dot x\dot\zeta)\frac{\partial F}{\partial\dot x}+(-\dot\varphi_x-\dot y\dot\zeta)\frac{\partial F}{\partial\dot y}+3\,\dot\zeta F,$$

$$\bar BF\equiv\psi_y\frac{\partial F}{\partial x}-\psi_x\frac{\partial F}{\partial y}+(\dot\psi_y-\dot x\dot\zeta)\frac{\partial F}{\partial\dot x}+(-\dot\psi_x-\dot y\dot\zeta)\frac{\partial F}{\partial\dot y}+3\dot\zeta F,$$

then the condition for $AF=0$ and $BF=0$ necessary and sufficient to have a common solution is that

$$\bar AB-\bar BA\equiv\lambda_1\,.\,A+\lambda_2\,.\,B.$$

But in our case,

$$\bar AB-\bar BA=\chi_y\frac{\partial F}{\partial x}-\chi_x\frac{\partial F}{\partial y}+(\dot\chi_y-\dot x\dot\rho)\frac{\partial F}{\partial\dot x}+(-\dot\chi_x-\dot y\dot\rho)\frac{\partial F}{\partial\dot y}$$
$$+3\,\dot\rho F-\left(\dot x\frac{\partial}{\partial x}+\dot y\frac{\partial}{\partial y}\right)^3\chi,$$

where $\chi \equiv \varphi_x \psi_y - \varphi_y \psi_x$ and $\rho \equiv \frac{1}{k}(2\chi - x\chi_x - y\chi_y)$; therefore it must be $\chi = \lambda_1 \varphi + \lambda_2 \psi + c$, where λ_1, λ_2 and c are any constants.

And also we know

$$(U_1, U_2) \equiv U_1(U_2 f) - U_2(U_1 f)$$

$$\equiv \chi_y \frac{\partial f}{\partial x} - \chi_x \frac{\partial f}{\partial y} + \rho \frac{\partial f}{\partial t},$$

so that we have

$$(U_1, U_2) \equiv \lambda_1 \, . \, U_1 f + \lambda_2 \, . \, U_2 f + 2c \frac{\partial f}{\partial t};$$

it shows that $U_1 f$, $U_2 f$ and $U_3 f \equiv \frac{\partial f}{\partial t}$ form the "3 gliedrige Gruppe." Similarly if we consider the two point-transformations

$$U_1 f \equiv \varphi_y \frac{\partial f}{\partial x} - \varphi_x \frac{\partial f}{\partial x}, \quad U_2 f \equiv \psi_y \frac{\partial f}{\partial x} - \psi_x \frac{\partial f}{\partial y},$$

then we have $(U_1, U_2) \equiv \lambda_1 \, . \, U_1 f + \lambda_2 \, . \, U_2 f$; it shows that $U_1 f$ and $U_2 f$ form the "2 gliedrige Gruppe."

So we have

Theorem 16: If the equations of motion

$$\ddot{x} = x \, . \, F(x, y, \dot{x}, \dot{y}), \; \ddot{y} = y \, . \, F(x, y, \dot{x}, \dot{y})$$

admit of two infinitesimal transformations

$$U_1 f \equiv \varphi_y \frac{\partial f}{\partial x} - \varphi_x \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial t}, \quad U_2 f \equiv \psi_y \frac{\partial f}{\partial x} - \psi_x \frac{\partial f}{\partial y} + \bar{\zeta} \frac{\partial f}{\partial t}.$$

then $U_1 f$, $U_2 f$, and $U_3 f \equiv \frac{\partial f}{\partial t}$ form "3 gliedrige Gruppe;" and there exists a relation between φ and ψ

$$\varphi_x \psi_y - \varphi_y \psi_x = \lambda_1 \varphi + \lambda_2 \psi + c,$$

where λ_1, λ_2 and c are any constants.

Also combining the above theorem with Theorem 12, we have

Theorem 17: If a system of the central motion admits of two infinitesimal point-transformations, then they form the "2 gliedrige Gruppe."

Using the theorems above stated, we may integrate the equations of motion. We separate the cases into the following four:

1. $\varphi_x\psi_y - \varphi_y\psi_x = \lambda_1\varphi + \lambda_2\psi + c$, where $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.
2. $\varphi_x\psi_y - \varphi_y\psi_x = \lambda_1\varphi + c$, where $\lambda_1 \neq 0$.
3. $\psi_x\varphi_y - \varphi_y\psi_x = c$, where $c \neq 0$.
4. $\varphi_x\psi_y - \varphi_y\psi_x = 0$.

1. In this case, as $(U_1, U_2) \equiv \lambda_1 . U_1 + \lambda_2 . U_2 + 2c\dfrac{\partial f}{\partial t}$ and $(\overline{U}_1, \overline{U}_2) \equiv \lambda_1 . \overline{U}_1 + \lambda_2 \overline{U}_2$, U_1 and U_2, $\overline{U}_1$ and $\overline{U}_2$ are not both interchangeable; and $\varphi =$ const. and $\psi =$ const. are the systems of the curves different from each other.

By Theorem 15. we have as particular solution of $AF=0$

$$\overline{F} \equiv -\frac{1}{k(\lambda_1\varphi + \lambda_2\psi + c)}\left\{\dot{\psi}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^2\varphi - \dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^2\psi\right\},$$

but it is easily seen that $\overline{F}$ also satisfies the second equation $BF=0$, and moreover that $\left\{\dfrac{d}{dt}\log(\lambda_1\varphi + \lambda_2\psi + c)\right\}^3$ satisfies both the equations $\overline{A}F=0$ and $\overline{B}F=0$.

So that the general solutions of $AF=0$ and $BF=0$ are respectively of the forms

$$F_1 \equiv \overline{F} + \left\{\frac{d}{dt}\log(\lambda_1\varphi + \lambda_2\psi + c)\right\}^3 . W\left(\varphi, \frac{d}{d\varphi}\log(\lambda_1\varphi + \lambda_2\psi + c)\right),$$

$$F_2 \equiv \overline{F} + \left\{\frac{d}{dt}\log(\lambda_1\varphi + \lambda_2\psi + c)\right\}^3 . W_2\left(\psi, \frac{d}{d\psi}\log(\lambda_1\varphi + \lambda_2\psi + c)\right),$$

while W_2 can be written as

$$\overline{W}_2\left(\psi, \frac{1}{\dfrac{d}{d\varphi}\log(\lambda_1\varphi + \lambda_2\psi + c)} - \varphi\right),$$

for, since W_2 is the solution of $\overline{B}F=0$, by changing the variables as

$$\xi = \varphi(x, y),\ \eta = \log(\lambda_1\varphi + \lambda_2\psi + c),$$

we have

$$(\lambda_1\varphi + \lambda_2\psi + c)\frac{\partial F}{\partial \xi} + \lambda_1\frac{\partial F}{\partial \eta} + (\lambda_1\dot{\varphi} + \lambda_2\dot{\psi} - \dot{\varphi}\dot{\zeta})\frac{\partial F}{\partial \dot{\xi}} - \dot{\eta}\dot{\zeta}\frac{\partial F}{\partial \dot{\eta}} = 0,$$

from which we have the solution

$$F=\overline{W}_2\left(\psi, \frac{1}{\frac{d}{d\varphi}\log(\lambda_1\varphi+\lambda_2\psi+c)}-\varphi\right).$$

Thus our proposition is proved. Therefore, the required common solution is

$$\overline{F}+\left\{\frac{d}{dt}\log(\lambda_1\varphi+\lambda_2\psi+c)\right\}^3 \cdot W\left(\frac{1}{\frac{d}{d\rho}\log(\lambda_1\varphi+\lambda_2\psi+c)}-\varphi\right),$$

and the equations of motion become

$$\frac{d^2\varphi}{d\eta^2}=\lambda_2 W\left(\frac{d\varphi}{d\eta}-\varphi\right)+\frac{d\varphi}{d\eta}.$$

If we put $\frac{d\varphi}{d\eta}-\varphi\equiv\overline{\varphi}$, then the above equation becomes

$$\frac{d\overline{\varphi}}{d\eta}=\lambda_2 W(\overline{\varphi});$$

hence we have $\int\frac{d\overline{\varphi}}{W(\overline{\varphi})}=\lambda_2\eta+a$, that is

$$\overline{\varphi}=f(\lambda_2\eta+a),$$

where $f(x)$ is the inverse function of $\int\frac{dx}{W(x)}$.

Hence we have $\frac{d\varphi}{d\eta}-\varphi=f(\lambda_2\eta+a)$, from which we have

$$\varphi=\int e^{\eta}f(\lambda_2\eta+a)\,d\eta+b.$$

So we have

Theorem 17: If the equations of motion

$$\ddot{x}=x\,.\,F(x, y. \dot{x}, \dot{y}),\ \ddot{y}=y\,.\,F(x, y, \dot{x}, \dot{y})$$

admit of two infinitesimal transformations

$$\varphi_y\frac{\partial f}{\partial x}-\varphi_x\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial t},\ \psi_y\frac{\partial f}{\partial x}-\psi_x\frac{\partial f}{\partial y}+\overline{\zeta}\frac{\partial f}{\partial t},$$

satisfying the relation

$$\varphi_x\psi_y-\varphi_y\psi_x=\lambda_1\varphi+\lambda_2\psi+c,$$

then F must be of the form

$$\left\{\dot\varphi\left(\dot x\frac{\partial}{\partial x}+\dot y\frac{\partial}{\partial y}\right)^2\psi-\dot\psi\left(\dot x\frac{\partial}{\partial x}+\dot y\frac{\partial}{\partial y}\right)^2\varphi\right\}\Big/k\,(\lambda_1\varphi+\lambda_2\psi+c)$$
$$+\left\{\frac{d}{dt}\log(\lambda_1\varphi+\lambda_2\psi+c)\right\}^3.\ W\left(\frac{1}{\dfrac{d}{d\varphi}\log(\lambda_1\,\varphi+\lambda_2\psi+c)}-\varphi\right),$$

and the curves of motion are obtained by two quadratures as $\varphi=\int e^\eta f(\lambda_2\eta+a)\,d\eta+b$, where $f(x)$ is the inverse function of $\int\frac{dx}{W(x)}$, and $\eta=\log(\lambda_1\varphi+\lambda_2\psi+c)$.

2. In this case, $\varphi_x\psi_y-\varphi_y\psi_x=\lambda_1\varphi+c$, U_1 and U_2, $\overline{U}_1$ and $\overline{U}_2$ are not both interchangeable, and $\varphi=$const. admit of U_2 but $\psi=$const. does not admit of U_1, and the two systems of curves $\varphi=$const. and $\psi=$const. are different from each other.

If we interchange φ with ψ and λ_1 with λ_2, and put $\lambda_1=0$ in the case 1, then our case becomes the special case of 1.

3. In this case, since $\varphi_x\psi_y-\varphi_y\psi_x=c$, *i. e.* $(U_1, U_2)\equiv 2\frac{\partial f}{\partial t}$, $(\overline{U}_1, \overline{U}_2)\equiv 0$, U_1 and U_2 are not interchangeable, but $\overline{U}_1$ and $\overline{U}_2$ are interchangeable; and the curves $\varphi=$const, and $\psi=$const. admit of $\overline{U}_2$ and $\overline{U}_1$ respectively, and they are different from each other.

By the remark 2 in Theorem 16, we have as the general solution of $AF=0$

$$F_1=\overline{F}+\dot\varphi^3.\ W_1\left(\varphi,\frac{\dot\psi}{\dot\varphi}\right);$$

but we can see that $\overline{F}$ also satisfies the second equation $BF=0$, hence the general solution of $BF=0$ must be of the form

$$F_2=\overline{F}+\dot\psi^3\ W_2\left(\psi,\frac{\dot\psi}{\dot\varphi}\right).$$

But $\dot\psi^3.\ W_2\left(\psi,\frac{\dot\psi}{\dot\varphi}\right)$ can be written as $\dot\varphi^3.\left(\frac{\dot\psi^3}{\dot\varphi^3}\right).\ W_2\left(\psi,\frac{\dot\psi}{\dot\varphi}\right)$, so the common solution of $AF=0$ and $BF=0$ may be put into the form

$$F=\overline{F}+\dot\varphi^3.\ W\left(\frac{\dot\psi}{\dot\varphi}\right).$$

Hence the equations of motion are reduced to

$$\frac{d^2\psi}{d\varphi^2}=ck\,W\left(\frac{d\psi}{d\varphi}\right),$$

from which we get $\psi=kc\int\Phi(\varphi+a)\,d\varphi+b$, where $\Phi(x)$ is the inverse function of $\int\frac{dx}{W(x)}$ and a, b are constants.

So we have

Theorem 18: If the equations of motion

$$\ddot{x}=x\,.\,F(x, y, \dot{x}, \dot{y}),\ \ \ddot{y}=y\,.\,F(x, y, \dot{x}, \dot{y})$$

admit of two infinitesimal transformations having a relation $\varphi_x\psi_y-\varphi_y\psi_x=c$, then F must be of the form

$$\left\{\dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi-\dot{\psi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\varphi\right\}\Big/kc+\dot{\varphi}^3\,.\,W\left(\frac{\dot{\psi}}{\dot{\varphi}}\right),$$

and the curves of motion are obtained by two quadratures as such

$$\psi=ck\int\Phi(\varphi+a)\,d\varphi+b,$$

where $\Phi(x)$ is the inverse function of $\int\frac{dx}{W(x)}$.

4. In this case, since $\varphi_x\psi_y-\varphi_y\psi_x=0$, *i.e.* $(U_1, U_2)\equiv 0$ and $(\overline{U}_1, \overline{U}_2)\equiv 0$, U_1, U_2 and $\overline{U}_1, \overline{U}_2$ are both interchangeable. And also we know that $\psi=f(\varphi)$, hence the two systems of curves $\varphi=$const. and $\psi=$const. are of the same system.

Now, we can introduce two functions ψ_1 and ψ_2, so that

$$\varphi_x\psi_{1y}-\varphi_y\psi_{1x}=1,\ \ \psi_x\psi_{2y}-\psi_y\psi_{2x}=1\,;$$

then the solutions of $AF=0$ and $BF=0$ are respectively

$$F_1\equiv\overline{F}_1+\dot{\varphi}^3\,.\,W_1\left(\varphi, \frac{\dot{\psi}_1}{\dot{\varphi}}\right),$$

$$F_2\equiv\overline{F}_2+\dot{\psi}^3\,.\,W_2\left(\psi, \frac{\dot{\psi}_2}{\dot{\psi}}\right);$$

where

$$\overline{F}_1 = \frac{1}{k}\left\{\dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi_1 - \dot{\psi}_1\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\varphi\right\},$$

$$\overline{F}_2 \equiv \frac{1}{k}\left\{\dot{\psi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi_2 - \dot{\psi}_2\left(\dot{x}\frac{\partial}{\partial y}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi\right\}.$$

But from $\varphi_x\psi_{1y}-\varphi_y\psi_{1x}=1$, $\psi_x\psi_{2y}-\psi_y\psi_{2x}=1$, and $\psi=f(\varphi)$, we can take

$$\psi_2 = \frac{\psi_1}{f'(\varphi)},$$

hence we have, substituting the value of ψ_2 into F_2,

$$\overline{F}_2 = \overline{F}_1 - \frac{1}{k}\left\{f''' \cdot \psi_2 + 3\frac{\dot{\psi}_2}{\dot{\psi}} f'' \cdot f'\right\}\dot{\varphi}^3. \tag{34}$$

And also we have $\dfrac{d\psi_1}{d\varphi} = f'^2 \cdot \dfrac{d\psi_2}{d\psi} + f'' \cdot \psi_2$; and, therefore, if we take, without loss of generality, as

$$W_1 \equiv -\frac{1}{k}\frac{f'''}{f''}\frac{d\psi_1}{d\varphi} + \overline{W}\left(\varphi, \frac{d\psi_1}{d\varphi}\right),$$

then

$$F_1 = \overline{F}_1 - \dot{\varphi}^3\left[\frac{1}{k}f''' \cdot \psi_2 + \frac{1}{k}\frac{f''' \cdot f'^2}{f''} \cdot \frac{d\psi_2}{d\psi} + \overline{W}\left(\varphi, \frac{d\psi_1}{d\varphi}\right)\right].$$

Substituting (34) into the above, we have

$$F_1 = \overline{F}_2 + \dot{\varphi}^3\left\{\frac{1}{k}\left[3f'' \cdot f' - \frac{f''' \cdot f'^2}{f''}\right]\frac{d\psi_2}{d\psi} + \overline{W}\left(\varphi, \frac{d\psi_1}{d\varphi}\right)\right\}.$$

Hence if we take $\overline{W} \equiv W(\varphi)$ and compare F_1 and F_2, then F_1 takes the form of F_2, so that the most general common solution of $AF=0$ and $BF=0$ is that

$$F = \overline{F}_1 - \dot{\varphi}^3\left\{\frac{1}{k}\frac{f'''(\varphi)}{f''(\varphi)}\frac{d\psi_1}{d\varphi} - W(\varphi)\right\}.$$

Then the equations of motion are reduced to, as in the former cases,

$$\frac{d^2\psi_1}{d\varphi^2} = -\frac{f'''(\varphi)}{f''(\varphi)}\frac{d\psi_1}{d\varphi} + kW(\varphi),$$

which can be solved by two quadratures as follows:

$$\psi_1=k\int\frac{\int f''(\varphi)\,W(\varphi)\,d\varphi}{f''(\varphi)}d\varphi+a\int\frac{d\varphi}{f''(\varphi)}+b.$$

So we have

Theorem 19: If the equations of motion

$$\ddot{x}=x\,.\,F(x,y,\dot{x},\dot{y})\,,\;\ddot{y}=y\,.\,F(x,y,\dot{x},\dot{y})$$

admit of two infinitesimal transformations, having the relation $\varphi_x\psi_y-\psi_x\varphi_y=0$, then F must be of the form

$$\frac{1}{k}\left\{\dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi_1-\dot{\psi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\varphi-\frac{f'''(\varphi)}{f''(\varphi)}\,.\,\dot{\psi}_1\,.\,\dot{\varphi}^2\right\}$$
$$+\dot{\varphi}^3\,.\,W(\varphi)\,;$$

and the equations of motion are solved by two quadratures as follows:

$$\psi_1=k\int\frac{\int f''(\varphi)\,W(\varphi)\,d\varphi}{f''(\varphi)}\,.\,d\varphi+a\int\frac{d\varphi}{f''(\varphi)}+b,$$

where ψ_1 is determined by $\varphi_x\psi_{1y}-\varphi_y\psi_{1x}=1$ and ψ and φ are connected by $\psi=f(\varphi)$.

Example 1. Consider the case when $\zeta=0$, *i. e.* $x\varphi_x+y\varphi_y=2\varphi$.

If we take ψ as such $\psi_x=-\dfrac{y}{2\varphi}$, $\psi_y=\dfrac{x}{2\varphi}$, then ψ satisfies the relation

$$\varphi_x\psi_y-\varphi_y\psi_x=1.$$

Hence, by Theorem 18, the equations of motion

$$\ddot{x}=x\,.\,F\left(\varphi,\frac{\dot{\psi}}{\dot{\varphi}}\right),\;\ddot{y}=y\,.\,F\left(\varphi\frac{\dot{\psi}}{\dot{\varphi}}\right)$$

admit of the transformation $\varphi_y\dfrac{\partial f}{\partial y}-\varphi_x\dfrac{\partial f}{\partial y}$ and they are reduced to the equation

$$\frac{d^2\psi}{d\varphi^2}=F\left(\varphi,\frac{d\psi}{d\varphi}\right).$$

But as $\dot{\psi}=\dfrac{x\dot{y}-y\dot{x}}{2\varphi}$, $F\left(\varphi,\dfrac{d\psi}{d\varphi}\right)$ may be written in the form $W(\varphi,\dot{\psi})$, so we have

Theorem 20: The equations of motion

$$\ddot{x}=x\,.\,F(\varphi,\dot{\varphi}),\ \ \ddot{y}=y\,.\,F(\varphi,\dot{\varphi}),$$

having the first integral $x\dot{y}-y\dot{x}=k$, are reduced to the form $\dfrac{d^2\psi}{d\varphi^2}=W\left(\varphi,\dfrac{d\psi}{d\varphi}\right)$, $\psi d\varphi=kdt$, where $\varphi(x,y)$ is the homogeneous function of the second degree and $\psi=\displaystyle\int\frac{xdy-ydx}{2\,\varphi}$.

Specially, let $\varphi=ax^2+2\,hxy+by^2$ and use the identity

$$\{ax\dot{x}+2\,h(\dot{x}y+\dot{y}x)+by\dot{y}\}^2\equiv(ax^2+2\,hxy+by^2)(a\dot{x}^2+2\,h\dot{x}\dot{y}+b\dot{y}^2)$$
$$+(h^2-ab)(x\dot{y}-y\dot{x})^2,$$

then we have

Corollary 1: The equations of motion

$$\ddot{x}=x\,.\,F(ax^2+2hxy+by^2,\ a\dot{x}^2+2h\dot{x}\dot{y}+b\dot{y}^2,\ ax\dot{x}+h(x\dot{y}+y\dot{x})+by\dot{y}),$$

$$\ddot{y}=y\,.\,F(ax^2+2\,hxy+by^2,\ a\dot{x}^2+2\,h\dot{x}\dot{y}+b\dot{y}^2,\ ax\dot{x}+h(x\dot{y}+y\dot{x})+by\dot{y})$$

are solved by the integration of a linear differential equation and a quadrature.

If $a=b$ and $h=0$, then we have

Corollary 2: If F is the function of distance from the centre and velocity, then the central curves are obtained by the integration of a linear differential equation and a quadrature.

Example 2. If we take the function in Theorem 14, *i. e.* the solution of the equation

$$(\varphi_x^2-\varphi_y^2)(\varphi_{xx}-\varphi_{yy})+4\,\varphi_x\varphi_y\varphi_{xy}=0,$$

and $\psi=\displaystyle\int\frac{1}{\varphi_x^2+\varphi_y^2}(\varphi_y dx-\varphi_x dy)$, then we have

$$\varphi_x\psi_x-\varphi_y\psi_x=1.$$

Hence we have

Theorem: The equations of the motion

$$\ddot{x}=x\,.\,\left\{\dot{\psi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\varphi-\dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)^2\psi\right.$$

$$+\dot{\varphi}^3 \cdot W(\varphi, \dot{\varphi}, \dot{\psi})\Big\} \Big/ -k,$$

$$\ddot{y} = y \cdot \left\{\dot{\psi}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^2 \varphi - \dot{\varphi}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)^2 \psi\right.$$

$$\left. + \dot{\varphi}^3 \cdot W(\varphi, \dot{\varphi}, \dot{\psi})\right\} \Big/ -k$$

are solved by the integration of a linear differential equation and a quadrature, where φ is the solution of the equation $(\varphi_x^2 - \varphi_y^2)(\varphi_{xx} - \varphi_{yy}) + 4\varphi_x\varphi_y\varphi_{xy} = 0$ and $\psi = \int \frac{\varphi_y dx - \varphi_x dy}{\varphi_x^2 + \varphi_y^2}$. And the system of the central curves orthogonal to $\varphi = \text{const.}$, admits of the infinitesimal transformation $\varphi_y \frac{\partial f}{\partial x} - \varphi_x \frac{\partial f}{\partial y}$.

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On a Certain Expansion of Analytic Function,

by

YOSHITOMO OKADA, Sendai.

In this note, we will prove a certain extended Taylor-Cauchy's theorem, and give some applications of the theorem.

1. We will first prove the following theorem[1]:

Theorem. Let $\{P_n(x)\}$ *be a given sequence of functions*

$$P_n(x)=x^n(1+a_1{}^{(n)}x+a_2{}^{(n)}x^2+\cdots\cdots), \qquad n=0,1,2,\cdots\cdots,$$

regular for $|x|<\rho$. *If*

$$0<R<\varliminf_{n=\infty}\left[b_1^{(n)}+\mathrm{Max}\left(1/\rho,\, b_{n+1}^{(0)}/b_n^{(0)},\, b_n^{(1)}/b_{n-1}^{(1)},\cdots\cdots, b_2^{(n-1)}/b_1^{(n-1)}\right)\right]^{-1},$$

$b_\nu^{(\mu)}$ *being any positive number not less than* $|a_\nu^{(\mu)}|$, *and if*

$$F_n(R)=1+|a_1^{(n)}|R+|a_2^{(n)}|R^2+\cdots\cdots\leqq Q,$$

Q *being a positive number independent of* n, *then any given function* $f(x)$, *regular for* $|x|<\rho$, *can be expanded uniformly and uniquely into the form*

$$f(x)=\alpha_0P_0(x)+\alpha_1P_1(x)+\alpha_2P_2(x)+\cdots\cdots, \qquad \textit{for } |x|\leqq R.$$

Proof: Put

$$M_n(\rho)=\mathrm{Max}\,(1/\rho,\, b_{n+1}^{(0)}/b_n^{(0)},\, b_n^{(1)}/b_{n-1}^{(1)}\cdots\cdots,\, b_2^{(n-1)}/b_1^{(n-1)}).$$

Then from

$$R<\varliminf_{n=\infty}\{b_1^{(n)}+M_n(\rho)\}^{-1},$$

it can be seen that there exists a positive number $r<\rho$ such that

$$R<\varliminf_{n=\infty}\{b_1^{(n)}+M_n(r)\}^{-1}\leqq\varliminf_{n=\infty}\{b_1^{(n)}+M_n(\rho)\}^{-1}.$$

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(1) Compare with G. D. Birkhoff, Comptes Rendus, 164, 1917, pp. 942–945; and also with S. Kakeya, Proc. of the Physico-Math. Soc. of Japan, 2, 1920, pp. 96–104.

And since $M_n(r) \geqq \dfrac{1}{r}$, we get

$$\underline{\lim}_{n=\infty} \{b_1^{(n)} + M_n(r)\}^{-1} \leqq r.$$

Consequently

$$R < r < \rho.$$

Now, for any given function $f(x)$ regular for $|x| < \rho$, determine $\alpha_0, \alpha_1, \alpha_2, \cdots\cdots, \alpha_n, \cdots\cdots$, such that

$$\left.\begin{aligned}
f(0) &= \alpha_0, \\
\frac{f'(0)}{1\,!} &= a_1^{(0)}\alpha_0 + \alpha_1, \\
\frac{f''(0)}{2\,!} &= a_2^{(0)}\alpha_0 + a_1^{(1)}\alpha_1 + \alpha_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{f^{(n)}(0)}{n\,!} &= a_n^{(0)}\alpha_0 + a_{n-1}^{(1)}\alpha_1 + a_{n-2}^{(2)}\alpha_2 + \cdots\cdots + \alpha_n, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \qquad (1)$$

hold good. Then we get

$$\alpha_n = (-1)^n \begin{vmatrix}
f(0) & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
\frac{f'(0)}{1\,!} & a_1^{(0)} & 1 & 0 & \cdots & \cdots & 0 \\
\frac{f''(0)}{2\,!} & a_2^{(0)} & a_1^{(1)} & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{f^{(n)}(0)}{n\,!} & a_n^{(0)} & a_{n-1}^{(1)} & a_{n-2}^{(2)} & \cdots & \cdots & a_1^{(n-1)}
\end{vmatrix},$$

$$n = 0, 1, 2, \cdots\cdots\cdots\cdots.$$

Therefore, since $f(x)$ is regular for $|x| \leqq r$, we have

$$\alpha_n = \frac{(-1)^n}{2\pi i} \int f(t)\, D_n(t)\, dt, \qquad n = 0, 1, 2, \cdots\cdots, \qquad (2)$$

where, as a path of the integration, the circle with the center $t = 0$ and the radius r is taken, and

$$D_n(t)=\begin{vmatrix} t^{-1} & 1 & 0 & 0\cdots\cdots\cdots 0 \\ t^{-2} & a_1^{(0)} & 1 & 0\cdots\cdots\cdots 0 \\ t^{-3} & a_2^{(0)} & a_1^{(1)} & 1\cdots\cdots\cdots 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ t^{-n} & a_{n-1}^{(0)} & a_{n-2}^{(1)} & a_{n-3}^{(2)}\cdots\cdots 1 \\ t^{-(n+1)} & a_n^{(0)} & a_{n-1}^{(1)} & a_{n-2}^{(2)}\cdots a_1^{(n-1)} \end{vmatrix}.$$

Now let the determinant obtained by replacing t and $a_\nu^{(\mu)}$ in $D_n(t)$ by r and $b_\nu^{(\mu)}$ be $D_n'(t)$, and denote the sum of absolute values of all the terms in the expansion of the determinant $D_n'(t)$ by $E_n(t)$. Then by the assumption for $b_\nu^{(n)}$ we have

$$|D_n(t)| \leqq E_n(r), \qquad \text{for } |t|=r. \tag{3}$$

Since

$$D_n'(r)=b_1^{(n-1)}D'_{n-1}(r)-\begin{vmatrix} r^{-1} & 1 & 0 & 0\cdots\cdots\cdots 0 \\ r^{-2} & b_1^{(0)} & 1 & 0\cdots\cdots\cdots 0 \\ r^{-3} & b_2^{(0)} & b_1^{(1)} & 1\cdots\cdots\cdots 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ r^{-(n-1)} & b_{n-2}^{(0)} & b_{n-3}^{(1)} & b_{n-4}^{(2)}\cdots\cdots 1 \\ r^{-(n+1)} & b_n^{(0)} & b_{n-1}^{(1)} & b_{n-2}^{(2)}\cdots b_2^{(n-2)} \end{vmatrix}, \tag{4}$$

and since $E_{n-1}(r)M_{n-1}(r)$ is not less than the sum of absolute values of all the terms in the expansion of the last determinant of (4), we get

$$E_n(r) \leqq (b_1^{(n-1)}+M_{n-1}(r))\,E_{n-1}(r), \qquad n=1, 2,\cdots\cdots, \tag{5}$$

whence

$$E_n(r) \leqq \frac{1}{r}(b_1^{(0)}+M_0(r))(b_1^{(1)}+M_1(r))\cdots\cdots(b_1^{(n-1)}+M_{n-1}(r)), \tag{6}$$

$$n=0, 1, 2,\cdots\cdots.$$

Therefore from (2), (3) and (6) we infer

$$|\alpha_n| \leqq M(r)\,(b_1^{(0)}+M_0(r))(b_1^{(1)}+M_1(r))\cdots\cdots(b_1^{(n-1)}+M_{n-1}(r)),$$

$$n=0, 1, 2,\cdots\cdots,$$

$M(r)$ standing for the maximum value of $|f(x)|$ for $|x|=r$.

Since

$$R < \varliminf_{n=\infty} \{b_1^{(n)} + M_n(r)\}^{-1},$$

the series $1 + \sum_{n=1}^{\infty} (b_1^{(0)} + M_0(r))(b_1^{(1)} + M_1(r)) \cdots\cdots (b_1^{(n-1)} + M_{n-1}(r)) R^n$ is convergent; and by the hypothesis there exists a positive number Q independent of n such that

$$\left| 1 + \sum_{\nu=1}^{\infty} a_\nu^{(n)} x^\nu \right| \leqq F_n(R) \leqq Q, \qquad \text{for } |x| \leqq R.$$

Therefore, since we have

$$\sum_{n=0}^{\infty} \left| \alpha_n P_n(x) \right| \leqq M(r)\, Q \left\{ 1 + \sum_{n=1}^{\infty} (b_1^{(0)} + M_0(r))\, (b_1^{(1)} + M_1(r)) \cdots\cdots \right.$$
$$\left. (b_1^{(n-1)} + M_{n-1}(r))\, R^n \right\}$$

for $|x| \leqq R$, we see that $\sum_{n=0}^{\infty} \alpha_n P_n(x)$ is uniformly convergent for $|x| \leqq R$.

On the other hand, from (1) and (2) we get

$$\left| \Sigma_n(x) - S_n(x) \right| \leqq \varepsilon_n^{(0)} |\alpha_0| + \varepsilon_{n-1}^{(1)} |\alpha_1| R + \cdots\cdots + \varepsilon_0^{(n)} |\alpha_n| R^n \qquad (7)$$

for $|x| \leqq R$, where

$$\Sigma_n(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \cdots\cdots + \alpha_n P_n(x),$$

$$S_n(x) = f(0) + \frac{f'(0)}{1!} x + \cdots\cdots + \frac{f^{(n)}(0)}{n!} x^n,$$

$$\varepsilon_\nu^{(\mu)} = |a_{\nu+1}^{(\mu)}| R^{\nu+1} + |a_{\nu+2}^{(\mu)}| R^{\nu+2} + \cdots\cdots.$$

Since the series

$$T = |\alpha_0| + |\alpha_1| R + |\alpha_2| R^2 + \cdots\cdots$$

is convergent, for any given positive number ε, there exists an index m such that

$$|\alpha_{\nu+1}| R^{\nu+1} + |\alpha_{\nu+2}| R^{\nu+2} + \cdots\cdots < \frac{\varepsilon}{2Q}, \quad \nu \geqq m;$$

and, since each of $1+|a_1^{(\mu)}|R+|a_2^{(\mu)}|R^2+\cdots\cdots$, $\mu=0, 1, 2,\cdots\cdots, m$, is convergent, we can find an index m' such that

$$\varepsilon_\nu^{(\mu)}=|a_{\nu+1}^{(\mu)}|R^{\nu+1}+|a_{\nu+2}^{(\mu)}|R^{\nu+2}+\cdots\cdots<\frac{\varepsilon}{2T},\ \mu=0, 1, 2,\cdots\cdots m,\quad \nu\geqq m'.$$

Therefore from (7), for $|x|\leqq R$,

$$|\Sigma_n(x)-S_n(x)|<\frac{\varepsilon}{2T}(|\alpha_0|+|\alpha_1|R+|\alpha_2|R^2+\cdots\cdots)$$
$$+Q(|\alpha_{m+1}|R^{m+1}+|\alpha_{m+2}|R^{m+2}+\cdots\cdots)$$
$$<\varepsilon,\qquad\qquad n\geqq m+m'.$$

Hence

$$\lim_{n=\infty}\{\Sigma_n(x)-S_n(x)\}=0$$

holds uniformly for $|x|\leqq R$.

Therefore we can conclude that the function can be expanded uniformly into the form

$$f(x)=\alpha_0 P_0(x)+\alpha_1 P_1(x)+\alpha_2 P_2(x)+\cdots\cdots,\qquad \text{for } |x|\leqq R,$$

since $\lim \Sigma_n(x)=\sum_{n=0}^{\infty}\alpha_n P_n(x)$ converges and $S_n(x)$ converges to $f(x)$ uniformly for $|x|\leqq R$.

Lastly, from the uniqueness of $\alpha_0, \alpha_1, \alpha_2,\cdots\cdots, \alpha_n,\cdots\cdots$ such that (1), which can be easily obtained from an expansion of $f(x)$ of the above form, can hold good, the uniqueness of this expansion is seen at once.

2. Now we proceed to give some applications of this theorem.

(i) Let

$$P(x)=1+a_1 x+a_2 x^2+\cdots\cdots$$

be regular for $|x|<\rho$, and put

$$P_0(x)=P(x),$$

$$P_n(x)=n!\int_0^x\int_0^x\cdots\cdots\int_0^x P(x)\,dx^n$$
$$=x^n\left\{1+\frac{n!\,a_1}{2.\,3.\cdots(n+1)}x+\frac{n!\,a_2}{3.\,4.\cdots(n+2)}x^2+\cdots\cdots\right\},$$
$$n=1, 2, 3,\cdots\cdots.$$

Then $P_n(x)$ are regular for $|x|<\rho$.

Now, for any positive number $R<\rho$, take a positive number ε such that

$$R<\rho-\varepsilon<\rho.$$

Then, since $P(x)$ is regular, there exists a positive integer m' such that

$$|a_m|<1/(\rho-\varepsilon)^m, \qquad m>m'.$$

Let us determine $b_\nu^{(\mu)}$ such that

$$b_\nu^{(\mu)}=\frac{\mu!\,b_\nu}{(\nu+1)(\nu+2)\cdots\cdots(\nu+\mu)},$$

in which each b_ν is any positive number not less than $|a_\nu|$, for $\nu=1, 2 \cdots\cdots, m'$, and $b_\nu=1/(\rho-\varepsilon)^\nu$ for $\nu>m'$. Then we can choose a positive integer $n'>m'$ such that

$$\operatorname{Max}\left(1/\rho, b_{n+1}^{(0)}/b_n^{(0)},\ b_n^{(1)}/b_{n-1}^{(1)}, \cdots\cdots, b_2^{(n-1)}/b_1^{(n-1)}\right)$$

$$=\operatorname{Max}\left(\frac{1}{\rho}, \frac{1}{\rho-\varepsilon}, \frac{n}{(n+1)(\rho-\varepsilon)}, \cdots\cdots, \frac{m'+2}{(n+1)(\rho-\varepsilon)}, \frac{(m'+1)\,b_{m+1}}{(n+1)\,b_m},\right.$$
$$\left.\cdots\cdots, \frac{2\,b_2}{(n+1)\,b_1}\right)=1/(\rho-\varepsilon), \qquad n>n'.$$

Hence

$$\varliminf_{n=\infty}\left[b_1^{(n)}+\operatorname{Max}\left(1/\rho, b_{n+1}^{(0)}/b_n^{(0)}, b_n^{(1)}/b_{n-1}^{(1)}, \cdots\cdots, b_2^{(n-1)}/b_1^{(n-1)}\right)\right]^{-1}$$
$$=\rho-\varepsilon>R,$$

since $$\lim_{n=\infty} b_1^{(n)}=\lim_{n=\infty} b_1/(n+1)=0.$$

Further it is obviously seen that

$$F_n(R)=1+\frac{n!\,|a_1|}{2.\,3\cdots\cdots(n+1)}R+\frac{n!\,|a_2|}{3.\,4\cdots\cdots(n+2)}R^2+\cdots\cdots\leqq Q,$$

Q being independent of n.

Therefore by the theorem we have the result(1): *For a given function*

$$P(x)=1+a_1x+a_2x^2+\cdots\cdots$$

(1) Compare with E. T. Whittaker, A course of modern analysis, 1902, p. 110.

regular for $|x|<\rho$, *any function* $f(x)$ *regular for* $|x|<\rho$ *can be expanded uniformly and uniquely into the form*

$$f(x)=\alpha_0 P_0(x)+\alpha_1 P_1(x)+\alpha_2 P_2(x)+\cdots\cdots, \qquad \textit{for } |x|\leqq R<\rho,$$

where

$$P_0(x)=P(x) \text{ and } P_n(x)=\int_0^x P_{n-1}(x)\,dx, \quad n=1, 2, 3,\cdots\cdots.$$

(ii). Let

$$P(x)=1+a_1x+a_2x^2+\cdots\cdots$$

be regular for $|x|<\rho$, and put

$$P_n(x)=x^n P(x), \qquad n=0, 1, 2,\cdots\cdots.$$

Then assuming that b_ν is any positive number not less than $|a_\nu|$ we get

$$\text{Max}\,(1/\rho, b^{(0)}_{n+1}/b^{(0)}_n, b^{(1)}_n/b^{(1)}_{n-1},\cdots\cdots, b^{(n-1)}_2/b^{(n-1)}_1)$$

$$=\text{Max}\,(1/\rho, b_{n+1}/b_n, b_n/b_{n-1},\cdots\cdots, b_2/b_1)$$

and

$$b^{(n)}_1=b_1.$$

Therefore by the theorem any function regular for $|x|<\rho$ can be expanded uniformly and uniquely into the form

$$f(x)=\alpha_0 P_0(x)+\alpha_1 P_1(x)+\alpha_2 P_2(x)+\cdots\cdots, \qquad \text{for } |x|\leqq R,$$

or

$$f(x)=P(x)\,\{\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots\cdots\}, \qquad \text{for } |x|\leqq R,$$

provided that

$$0<R<\varliminf_{n=\infty}\{b_1+\text{Max}\,(1/\rho, b_{n+1}/b_n, b_n/b_{n-1},\cdots\cdots, b_2/b_1)\}^{-1}.$$

Now, since we can choose $f(x)$ to be a function having no zero point within the circle $|x|=\rho$, we can conclude that *any function*

$$P(x)=1+a_1x+a_2x^2+\cdots\cdots$$

regular for $|x|<\rho$ *has no root within the circle*

$$|x|=\varliminf_{n=\infty}\{b_1+\text{Max}\,(1/\rho, b_{n+1}/b_n, b_n/b_{n-1},\cdots\cdots, b_2/b_1)\}^{-1},$$

assuming for b's as before.

Specially if we take $P(x)$ to be a polynomial

$$P(x)=1+a_1x+a_2x^2+\cdots\cdots+a_nx^n,$$

then since we can take ρ arbitrarily large and we can make b_{n+1}/b_n, $b_{n+2}/b_{n+1},\cdots\cdots$ be arbitrarily small, we have the result: *Any polynomial*

$$P(x)=1+a_1x+a_2x^2+\cdots\cdots+a_nx^n$$

has no root within the circle

$$|x|=\operatorname*{Min}_{1\leqq k\leqq n-1}\{b_1+b_{k+1}/b_k\}^{-1}.$$

(iii) Lastly let us take

$$P_n(x)=2^n n!\,J_n(x)$$
$$=x^n\left\{1-\frac{x^2}{2^2.\,1!\,(n+1)}+\frac{x^4}{2^4.\,2!\,(n+1)(n+1)}-\cdots\cdots\right\}.$$

Now, determine $b_\nu^{(\mu)}$ such that

$$b_{2\nu}^{(n)}=\{2^{2\nu}.\,\nu!\,(n+1)(n+2)\cdots\cdots(n+\nu)\}^{-1}.$$
$$b_{2\nu-1}^{(n)}=\sqrt{b_{2\nu-2}^{(n)}\,b_{2\nu}^{(n)}},\ (b_0^{(n)}=1)\ \ \nu=1,2,3,\cdots\cdots.$$

Then we have

$$\frac{b_{n+1}^{(0)}}{b_n^{(0)}}=\frac{1}{2\sqrt{(\nu+1)^2}} \qquad \text{if } n=2\nu,$$
$$=\frac{1}{2\sqrt{\nu^2}} \qquad \text{if } n=2\nu-1,$$
$$\frac{b_n^{(1)}}{b_{n-1}^{(1)}}=\frac{1}{2\sqrt{\nu(\nu+1)}} \qquad \text{if } n=2\nu,$$
$$=\frac{1}{2\sqrt{\nu(\nu+1)}} \qquad \text{if } n=2\nu-1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{b_3^{(n-2)}}{b_2^{(n-2)}}=\frac{1}{2\sqrt{2n}},\qquad \frac{b_2^{(n-1)}}{b_1^{(n-1)}}=\frac{1}{2\sqrt{n}};$$

so that, putting

$$b_{m+1}^{(n-m)}/b_m^{(n-m)}=1/(2\sqrt{pq}), \qquad m=n,\ n-1,\cdots\cdots,\ 1,$$

we see that

$$p+q=n+2 \qquad \text{if } m \text{ is even,}$$

$$=n+1 \qquad \text{if } m \text{ is odd.}$$

Therefore

$$\operatorname{Max}\left(b_{2\nu+1}^{(n-2\nu)}/b_{2\nu}^{(n-2\nu)},\ b_{2\nu-1}^{(n-2\nu+2)}/b_{2\nu-2}^{(n-2\nu+2)},\cdots\cdots, b_{3}^{(n-2)}/b_{2}^{(n-2)}\right)=1/(2\sqrt{2n})$$

and

$$\operatorname{Max}\left(b_{2\nu}^{(n-2\nu+1)}/b_{2\nu-1}^{(n-2\nu+1)},\ b_{2\nu-2}^{(n-2\nu+3)}/b_{2\nu-3}^{(n-2\nu+3)},\ \cdots\cdots, b_{2}^{(n-1)}/b_{1}^{(n-1)}\right)=1/(2\sqrt{n}),$$

consequently

$$\operatorname{Max}\left(1/\rho, b_{n+1}^{(0)}/b_{n}^{(0)}, b_{n}^{(1)}/b_{n-1}^{(1)},\cdots\cdots, b_{2}^{(n-1)}/b_{1}^{(n-1)}\right)=\operatorname{Max}\left(1/\rho, 1/(2\sqrt{n})\right).$$

Hence

$$\varliminf_{n=\infty}\left\{b_{1}^{(n)}+\operatorname{Max}\left(1/\rho, b_{n+1}^{(0)}/b_{n}^{(0)}, b_{n}^{(1)}/b_{n-1}^{(1)},\cdots\cdots, b_{2}^{(n-1)}/b_{1}^{(n-1)}\right)\right\}^{-1}$$

$$=\lim_{n=\infty}\left\{1/(2\sqrt{n+1})+\operatorname{Max}\left(1/\rho, 1/(2\sqrt{n})\right)\right\}^{-1}=\rho,$$

since

$$b_{1}^{(n)}=\sqrt{b_{2}^{(n)}}=1/(2\sqrt{n+1}).$$

Further it is obviously seen that

$$F_n(R)=1+\frac{R^2}{2^2 1!\,(n+1)}+\cdots\cdots\leqq Q, \quad \text{for } R<\rho,$$

Q being independent of n.

Hence by the theorem we see that a function $f(x)$ regular for $|x|<\rho$ can be expanded uniformly and uniquely into the form

$$f(x)=\alpha_0 P_0(x)+\alpha_1 P_1(x)+\alpha_2 P_2(x)+\cdots\cdots, \qquad \text{for } |x|\leqq R<\rho. \quad (8)$$

Therefore, putting $P_n(x)=2^n n!\,J_n(x)$ and $\alpha_n=\dfrac{C_n}{2^n\,n!}$ into (8), we get the following result which is nothing but the Neumann's expansion by the Bessel functions: *Any function $f(x)$ regular for $|x|<\rho$ can be expanded uniformly and uniquely into the form*

$$f(x)=C_0 J_0(x)+C_1 J_1(x)+C_2 J_2(x)+\cdots\cdots, \qquad \textit{for } |x|\leqq R<\rho.$$

Now, specially let $f(x)$ be a regular and even function for $|x|<\rho$. Then, since $f^{(2n+1)}(0)=0$ we can get from (1)

$$C_{2n}=2^{2n}(2n)!\ \alpha_{2n}$$

$$=(-1)^n 2^{2n}(2n)!$$

$$\cdot\begin{vmatrix} f(0) & 1 & 0 \cdots\cdots\cdots & 0 \\ \frac{f''(0)}{2!} & \frac{-1}{2^2 1!^2} & 1 \cdots\cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots \\ \frac{f^{(2n)}(0)}{(2n)!} & \frac{(-1)^n}{2^{2n} n!^2} & \frac{(-1)^{n-1}2!}{2^{2(n-1)}(n-1)!(n+1)!} \cdots & \frac{-(2n-2)!}{2^2 1!(2n-1)!} \end{vmatrix} \tag{9}$$

and $C_{2n+1}=0.$

Similarly let $f(x)$ be a regular and odd function for $|x|<\rho$. Then we have

$$C_{2n}=0$$

and $$C_{2n+1}=2^{2n+1}(2n+1)!\ \alpha_{2n+1}$$

$$=(-1)^n 2^{2n+1}(2n+1)!$$

$$\cdot\begin{vmatrix} \frac{f'(0)}{1!} & 1 & 0 \cdots\cdots\cdots & 0 \\ \frac{f'''(0)}{3!} & \frac{-1}{2^2 1!2!} & 1 \cdots\cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots \\ \frac{f^{(2n+1)}(0)}{(2n+1)!} & \frac{(-1)^n 1!}{2^{2n} n!(n+1)!} & \frac{(-1)^{n-1}3!}{2^{2(n-1)}(n-1)!(n+1)!} \cdots & \frac{-(2n-1)!}{2^2 1!(2n)!} \end{vmatrix} . \tag{10}$$

Hence *any function $f(x)$ regular and even, or odd, for $|x|<\rho$ can be expanded uniformly and uniquely into the form*

$$f(x)=C_0J_0(x)+C_2J_2(x)+C_4J_4(x)+\cdots\cdots, \textit{ for } |x|\leqq R<\rho,$$

or

$$f(x)=C_1J_1(x)+C_3J_3(x)+C_5J_5(x)+\cdots\cdots, \textit{ for } |x|\leqq R<\rho.$$

For examples, consider the functions

$$f(x)=\cos x \text{ and } f(x)=\sin x.$$

Then, for $f(x)=\cos x$, from (9) we have

$$c_0=1,\ c_2=-2,\ c_4=2,\cdots\cdots;$$

and, for $f(x)=\sin x$, from (10), we get

$$c_1=2,\ c_3=-2,\ c_5=2,\cdots\cdots$$

Therefore we obtain

$$\cos x=J_0(x)-2\,J_2(x)+2\,J_4(x)-\cdots\cdots$$

and

$$\sin x=2\,J_1(x)-2\,J_3(x)+2\,J_5(x)-\cdots\cdots,$$

which are due to Neumann.

May 1922.

Eine Verallgemeinerung des Taylor-Cauchyschen Satzes,

von

TADAHIKO KUBOTA in Sendai.

In der letzten Sitzung der hiesigen mathematischen Gesellschaft hat Herr Okada eine Darstellbarkeitsbedingung der analytischen Funktion $f(x)$ durch die Reihe von der Form

$$a_0 P(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots\cdots + a_n P_n(x) + \cdots\cdots,$$

wobei

$$P_{n+1}(x) = \int_0^x P_n(x)\, dx$$

$$P_1(x) = \int_0^x P(x)\, dx, \quad P(0) \neq 0$$

vorgelegt. Im Anschluss daran möchte ich hier seinen Satz auf einem anderen Wege beweisen und auf die Funktionen mehrerer Variabeln erweitern.

I. *Es sei $P(x)$ eine reguläre analytische Funktion von x im Kreise $|x| \leqq R$, ferner seien*

$$P(0) \neq 0, \quad P_1(x) = \int_0^x P(x)\, dx,$$

$$P_{n+1}(x) = \int_0^x P_n(x)\, dx\,;$$

wenn nun $f(x)$ im Kreise $|x| \leqq R$ regulär analytisch ist, dann ist $f(x)$ durch eine gleichmässig konvergente Reihe von der Form

$$a_0 P(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots\cdots + a_n P_n(x) + \cdots\cdots$$

darstellbar, wobei a_i geeignet zu wählen sind(1).

Der obige Satz lässt sich natürlicherweise in nachstehender Form ausdrücken:

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(1) Für die Bestimmung von a_i vergleiche man Whittaker, A course of Modern Analysis, Cambr. 1902.

II. Es sei $P(x)$ eine reguläre analytische Funktion von x im Kreise $|x|<R$, ferner seien

$$P(0)\neq 0, \qquad P_1(x)=\int_0^x P(x)\,dx,$$

$$P_{n+1}(x)=\int_0^x P_n(x)\,dx;$$

wenn nun $f(x)$ im Kreise $|x|<R$ regulär analytisch ist, dann ist $f(x)$ im Kreise $|x|<R$ durch eine konvergente Reihe von der Form

$$a_0P(x)+a_1P_1(x)+a_2P_2(x)+\cdots\cdots+a_nP_n(x)+\cdots\cdots$$

darstellbar und zwar ist die Konvergenz in jedem innerhalb des Kreises $|x| < R$ enthaltenen Bereiche gleichmässig.

Beweis des Satzes I.

Aus

$$P_{n+1}(x)=\int_0^x\int_0^x\cdots\cdots\int_0^x P(x)\,dx\,dx\cdots\cdots dx$$

erhält man in bekannter Weise

$$P_{n+1}(x)=\frac{1}{n!}\int_0^x (x-\xi)^n P(\xi)\,d\xi.$$

Wenn $f(x)$ also durch eine gleichmässig konvergente Reihe von der Form:

$$a_0P(x)+a_1P_1(x)+a_2P_2(x)+\cdots\cdots+a_nP_n(x)+\cdots\cdots$$

darstellbar ist, so ist

$$f(x)=a_0P(x)+\sum_{n=1}^{\infty}\int_0^x \frac{a_{n+1}}{n!}(x-\xi)^n P(\xi)\,d\xi,$$

wobei a_0 so zu bestimmen ist, dass

$$f(0)-a_0P(0)=0$$

gilt, d. h.

$$f(x)-\frac{f(0)}{P(0)}P(x)=\int_0^x P(\xi)\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}(x-\xi)^n\,d\xi.$$

Setzt man nun $x-\xi=t$, so ist

$$f(x)-\frac{f(0)}{P(0)}P(x)=\int_0^x P(x-t)\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}t^n\,dt.$$

Setzt man ferner

$$\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}t^n=F(t),$$

so ist

$$f(x)-\frac{f(0)}{P(0)}P(x)=\int_0^x P(x-t)F(t)\,dt.$$

Wir fragen uns ob man eine in Kreise $|x|\leqq R$ reguläre analytische Funktion $F(x)$ in solcher Weise bestimmen kann, dass die obige Relation gilt. Differenziert man die obige Relation nach x, so bekommt man

$$f'(x)-\frac{f(0)}{P(0)}P'(x)=P(0)F(x)+\int_0^x P'(x-t)F(t)\,dt,$$

d. h. $$\frac{f'(x)P(0)-f(0)P'(x)}{P(0)^2}=F(x)+\int_0^x\frac{P'(x-t)}{P(0)}F(t)\,dt,$$

welche eine Volterra'sche Integralgleichung zweiter Art ist.

Nun sei der grösste absolute Betrag von $\left|\frac{P'(x)}{P(0)}\right|$ im Kreise $|x|\leqq R$ mit M bezeichnet. Ferner sei der grösste absolute Betrag von

$$\frac{f'(x)P(0)-f(0)P'(x)}{P(0)^2}$$

im Kreise $|x|\leqq R$ mit N bezeichnet.

Nun setze man

$$\frac{f'(x)P(0)-f(0)P'(x)}{P(0)^2}=\varphi(x),$$

$$\frac{P'(x-t)}{P(0)}=-K(x,t),$$

dann ist

$$\varphi(x)=F(x)-\int_0^x K(x,t)F(t)\,dt.$$

Berechnet man

$$K_1(x, t) = \int_t^x K(x, y)\, K(y, t)\, dy,$$

$$K_p(x, t) = \int_t^x K(x, y)\, K_{p-1}(y, t)\, dy,$$

dann wird die Lösung der Integralgleichung

$$\varphi(x) = F(x) - \int_0^x K(x, t)\, F(t)\, dt$$

durch

$$F(x) = \varphi(x) + \int_0^x \mathfrak{K}(x, t\,;\, 1)\, \varphi(t)\, dt$$

darstellbar, wobei

$$\mathfrak{K}(x, t\,;\, \lambda) = K(x, t) + \lambda K_1(x, t) + \lambda^2 K_2(x, t) + \cdots\cdots$$
$$+ \lambda^n K_n(x, t) + \cdots\cdots, \quad |\, K_n(x, t)\,| \leqq \frac{M^{n+1} R^n}{n!}.$$

Diese Reihe $\mathfrak{K}(x, y\,;\, 1)$ konvergiert bekanntlich in jedem endlichen Bereich, wo K stetig ist, gleichmässig. Folglich stellt $\mathfrak{K}(x, t\,;\, 1)$ in unserem Falle eine reguläre analytische Funktion im Bereiche $|\,x\,| \leqq R$, $|\,t\,| \leqq R$, $|\,x - t\,| < R$ dar.

Infolgedessen ist die Lösung $F(t)$ regulär analytisch im Kreise $|\,t\,| \leqq R$ und ist durch eine gleichmässig konvergente Reihe

$$a_1 + \frac{a_2 t}{1!} + \frac{a_3 t^2}{2!} + \frac{a_4 t^3}{3!} + \cdots\cdots + \frac{a_{n+1} t^n}{n!} + \cdots\cdots$$

darstellbar. Somit ist die Reihe

$$a_0 P(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots\cdots + a_n P_n(x) + \cdots\cdots$$

im Bereiche $|\,x\,| \leqq R$ gleichmässig konvergent und stellt die gegebene Funktion $f(x)$ dar. Damit ist der vorgelegte Satz bewiesen.

Als unmittelbare Folge des obigen Satzes erhält man den folgenden

Satz III. *Es seien $g(x)$ und $f(x)$ im Bereiche $|\,x\,| < R$ regulär analytisch und man betrachte*

$$P_1(x)=\int_0^x P(x)g(x)dx,$$

$$P_{n+1}(x)=\int_0^x P_n(x)g(x)\,dx, \qquad P(0)\neq 0.$$

Bezeichnet man den kleinsten absoluten Betrag der singulären Punkte von der Umkehrung von

$$z=\int_0^x g(x)\,dx$$

mit k, *so ist* $f(x)$ *durch*

$$a_0P(x)+a_1P_1(x)+a_2P_2(x)+\cdots\cdots+a_nP_n(x)+\cdots\cdots$$

im gemeinsamen Bereich

$$\left|\int_0^x g(x)\,dx\right| < k \text{ und } |x| < R$$

darstellbar.

Zum Beispiel nehme man

$$g(x)=1+x, \qquad R=1,$$

$$\int_0^x g(x)\,dx=x+\frac{x^2}{2}=z,$$

$$x^2+2x-2z=0, \qquad x=-1\pm\sqrt{1+2z}.$$

Somit ist der kleinste absolute Betrag von singulären Punkten von der Umkehrung von

$$z=x+\frac{x^2}{2}$$

ist $\frac{1}{2}$. So ist der gesuchte Gültigkeitsbereich der gemeinsame Bereich von

$$|x(x+2)| < 1, \qquad |x| < 1.$$

Nun wollen wir unsere Betrachtung auf die analytischen Funktionen mehrerer Veränderlichen verallgemeinern:

IV. *Es sei* $P(x, y)$ *im Bereiche*

$$|x| \leqq R, \qquad |y| \leqq \rho$$

regulär analytisch, ferner seien

$$P(0,0) \neq 0,$$

$$P_{10}(x,y) = \int_0^x P(x,y)\,dx,$$

$$P_{01}(x,y) = \int_0^y P(x,y)\,dy,$$

$$P_{20}(x,y) = \int_0^x \int_0^x P(x,y)\,dx\,dx,$$

$$P_{11}(x,y) = \int_0^x \int_0^y P(x,y)\,dx\,dy,$$

$$P_{02}(x,y) = \int_0^y \int_0^y P(x,y)\,dy\,dy, \quad \text{u. s. w.},$$

dann ist jede im Bereiche $|x| \leqq R$, $|y| \leqq \rho$ *reguläre analytische Funktion* $f(x,y)$ *in demselben Bereiche durch eine gleichmässig konvergente Reihe von der Form*

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} P_{mn}(x,y)$$

darstellbar.

Beweis.

Nach der Definition von $P_{m,n}$ erhält man:

$$P_{m,n}(x,y) = \int_0^x \int_0^y P(u,v) \frac{(x-u)^{m-1}}{(m-1)!} \frac{(y-v)^{n-1}}{(n-1)!}\,du dv, \; m>0, n>0,$$

$$P_{m,0}(x,y) = \int_0^x P(u,y) \frac{(x-u)^{m-1}}{(m-1)!}\,du,$$

$$P_{0,n}(x,y) = \int_0^y P(x,v) \frac{(y-v)^{n-1}}{(n-1)!}\,dv.$$

Nun setze man

$$f(x,y) - \frac{f(0,0)}{P(0,0)} P(x,y) = \varphi(x,y)$$

und dann bestimme man $a_{i,0}$ $(i=1, 2, 3 \cdots\cdots \infty)$ sodass

$$\varphi(x, 0)=\sum_{m=1}^{\infty} a_{m0}\int_0^x P(u, 0)\,\frac{(x-u)^{m-1}}{(m-1)!}\,du$$

$$=\int_0^x \sum_{m=0}^{\infty} a_{m0}\,\frac{(x-u)^{m-1}}{(m-1)!}\,P(u,0)\,du.$$

Setzt man nun

$$F(t)=\sum_{m=0}^{\infty} a_{m+1}\,\frac{t^m}{m!},$$

so ist

$$\varphi(x, 0)=\int_0^x P(u, 0)\,F(x-u)\,du$$

$$=\int_0^x P(x-t, 0)\,F(t)\,dt.$$

Aus der obigen Integralgleichung kann man $F(t)$, d. h. $a_{i,0}$ $(i=1, 2, 3\cdots\cdots\infty)$ bestimmen.

Nun
$$\varphi(0, y)=\sum_{n=1}^{\infty} a_{0n}\int_0^y P(0, v)\,\frac{(y-v)^{n-1}}{(n-1)!}\,dv$$

$$=\int_0^y \sum_{n=1}^{\infty} a_{0n}\,\frac{(y-v)^{n-1}}{(n-1)!}\,P(0, v)\,dv.$$

Setzt man

$$\sum_{n=1}^{\infty} a_{0n}\,\frac{\tau^{n-1}}{(n-1)!}=G(\tau),$$

so erhält man

$$\varphi(0, y)=\int_0^y P(0, v)\,G(y-v)\,dv$$

$$=\int_0^y P(0, y-\tau)\,G(\tau)\,d\tau.$$

Bestimmt man $G(\tau)$ daraus, so sind a_{0n} $(n=1, 2, 3\cdots\cdots\infty)$ bestimmbar. Nun setze man

$$\varphi(x, y)-\int_0^x P(x-t, y)\,F(t)\,dt-\int_0^y P(x, y-\tau)\,G(\tau)\,d\tau=\psi(x, y),$$

so gelten $\psi(0, y) \equiv 0,\ \psi(x, 0) \equiv 0.$

Dann setze man

$$\psi(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} P_{mn}(x, y)$$

$$= \int_0^y \int_0^x P(u, v) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m+1,\, n+1} \frac{(x-u)^m}{m\,!} \frac{(y-v)^n}{n\,!}\, du\, dv.$$

Setzt man ferner

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m+1,\, n+1} \frac{(x-u)^m}{m\,!} \frac{(y-v)^n}{n\,!} = \Phi(x-u, y-v),$$

so erhält man

$$\psi(x, y) = \int_0^y \int_0^x P(u, v)\ \Phi(x-u, y-v)\, du dv$$

$$= \int_0^y \int_0^x P(x-t, y-\tau)\ \Phi(t, \tau)\, dt\, d\tau, \tag{1}$$

wobei $\psi(0, y) \equiv 0, \qquad \psi(x, 0) \equiv 0.$

Differenziert man (1) nach x und nach y

$$\frac{\partial^2 \psi(x, y)}{\partial x\, \partial y} = P(0, 0)\ \Phi(x, y) + \int_0^y P_y(0, y-\tau)\, \Phi(x, \tau)\, d\tau$$

$$+ \int_0^x P_x(x-t, 0)\, \Phi(t, y)\, dt + \int_0^x \int_0^y P_{xy}(x-t, y-\tau)\ \Phi(t, \tau)\, dt\, d\tau. \tag{2}$$

Umgekehrt kann man (1) aus (2) wie folgt herleiten: Integriert man (2), so ergibt sich

$$\psi(x, y) + X(x) + Y(y) = \int_0^y \int_0^x P(x-t, y-\tau)\ \Phi(t, \tau)\, dt\, d\tau,$$

wobei $X(x)$, $Y(y)$ analytische Funktionen bez. von x und von y sind. Setzt man $x=0$, $y=0$ der Reihe nach

$$\psi(0, y) + X(0) + Y(y) \equiv 0,$$

$$\psi(x, 0) + X(x) + Y(0) \equiv 0,$$

$$Y(y) \equiv -X(0), \ X(x) \equiv -Y(0), \ X(0)+Y(0)=0,$$

folglich
$$X(x)+Y(y) \equiv 0.$$

Die Integralgleichung (2) ist von der Form

$$\varphi(x,y)=\Phi(x,y)+\int_0^y K(y,\tau)\,\Phi(x,\tau)\,d\tau+\int_0^x L(x,t)\,\Phi(t,y)\,dt$$

$$+\int_0^x\int_0^y M(x,t\,;\,y,\tau)\,\Phi(t,\tau)\,dt\,d\tau\,(^1),$$

wobei die Funktion $\Phi(x,y)$ die gesuchte Funktion darstellt.

In den Bereichen $|x| \leqq R$, $|t| \leqq R$, $|x-t| \leqq R$, $|y| \leqq \rho$, $|\tau| \leqq \rho$, $|y-\tau| \leqq \rho$ seien

$$|\varphi(x,y)|<G, \ |K(y,\tau)|<N, \ |L(x,t)|<N,$$

$$|M(x,t\,;\,y,\tau)|<N.$$

Wir finden nun $\Phi(x,y)$ durch die Method der aufeinanderfolgenden Annäherung nämlich

$$\Phi_1(x,y) \equiv \varphi(x,y),$$

$$\Phi_2(x,y)=\varphi(x,y)-\int_0^y K(y,\tau)\,\varphi(x,\tau)\,d\tau$$

$$-\int_0^x L(x,t)\,\varphi(t,y)\,dt-\int_0^x\int_0^y M(x,t\,;\,y,\tau)\,\varphi(t,\tau)\,dt\,d\tau,$$

$$\Phi_3(x,y)=\varphi(x,y)-\int_0^y K(y,\tau)\,\varphi(x,\tau)\,d\tau$$

$$-\int_0^x L(x,t)\,\varphi(t,y)\,dt-\int_0^x\int_0^y M(x,t\,;\,y,\tau)\,\varphi(t,\tau)\,dt\,d\tau$$

$$+\int_0^y K(y,\bar\tau)\left[\int_0^{\bar\tau} K(\bar\tau,\tau)\,\varphi(x,\tau)\,d\tau\right]d\bar\tau$$

$$+\int_0^x L(x,\bar t)\left[\int_0^y K(y,\tau)\,\varphi(\bar t\,\tau)\,d\tau\right]d\bar t$$

$$+\int_0^x\int_0^y M(x,\bar t\,;\,y,\bar\tau)\left[\int_0^\tau K(\bar\tau,\tau)\,\varphi(\bar t\,\tau)\,d\tau\right]d\bar t\,d\bar\tau$$

(1) Vgl. Volterra, Leçons sur les équations intégrales, s. 74-77, darin ist die Aufgabe kurz behandelt.

$$+\int_0^y K(y,\bar\tau)\left[\int_0^x L(x,t)\,\varphi(t,\bar\tau)\,dt\right]d\bar\tau$$

$$+\int_0^x L(x,\bar t)\left[\int_0^{\bar t} L(\bar t,t)\,\varphi(t,y)\,dt\right]d\bar t$$

$$+\int_0^x\int_0^y M(x,\bar t;\ y\,\bar\tau)\left[\int_0^{\bar t} L(\bar t,t)\,\varphi(t,\bar\tau)\,dt\right]d\bar t\,d\bar\tau$$

$$+\int_0^y K(y,\bar\tau)\left[\int_0^x\int_0^{\bar\tau} M(x,t;\ \bar\tau\,\tau)\,\varphi(t,\tau)\,dt\,d\tau\right]d\bar\tau$$

$$+\int_0^x L(x,\bar t)\left[\int_0^{\bar t}\int_0^y M(\bar t,t;y,\tau)\,\varphi(t,\tau)\,dt\,d\tau\right]d\bar t$$

$$+\int_0^x\int_0^y M(x,\bar t;\ y,\bar\tau)\left[\int_0^{\bar t}\int_0^{\bar\tau} M(\bar t,t;\ \bar\tau,\tau)\,\varphi(t,\tau)\,dt\,d\tau\right]d\bar t\,d\bar\tau.$$

Zum Beispiel

$$\left|\int_0^y K(y,\bar\tau)\left[\int_0^{\bar\tau} K(\bar\tau,\tau)\,\varphi(x,\tau)\,d\tau\right]d\bar\tau\right|$$

$$=\left|\int_0^y \varphi(x,\tau)\left[\int_\tau^y K(y,\bar\tau)\,K(\bar\tau,\tau)\,d\bar\tau\right]d\tau\right|$$

$$\leqq G\,N^2\,\frac{|y|^2}{2\,!}\leqq G\,N^2\,\frac{\rho^2}{2\,!}$$

$$\left|\int_0^x L(x,\bar t)\left[\int_0^y K(y,\tau)\,\varphi(\bar t,\tau)\,d\tau\right]d\bar t\right|$$

$$\leqq G\,N^2\,|x|\,\,|y|\leqq G\,N^2\,R\,\rho.$$

$$\left|\int_0^x\int_0^y M(x,\bar t;\ y,\bar\tau)\left[\int_0^{\bar\tau} K(\bar\tau,\tau)\,\varphi(\bar t,\tau)\,d\tau\right]d\bar t\,d\bar\tau\right|$$

$$\leqq G\,N^2\,\frac{|x|\,|y|^2}{2\,!}\leqq G\,N^2\,\frac{R\,\rho^2}{2\,!}\qquad \text{u. s. w.}$$

Daraus erhält man im allgemeinen

$$|\,\Phi_{n+2}(x,y)-\Phi_{n+1}(x,y)\,|$$

$$\leqq G\,N^{n+1}\,\frac{\rho^{n+1}}{(n+1)\,!}+{}_{n+1}C_1\,G\,N^{n+1}\,\frac{R\,\rho^n}{n\,!}+{}_{n+1}C_2\,G\,N^{n+1}\,\frac{R^2\,\rho^{n-1}}{2\,!\,(n-1)\,!}$$

$$+\cdots\cdots+{}_{n+1}C_r\,G\,N^{n+1}\frac{R^r\rho^{n-r+1}}{r!\,(n-r+1)!}+\cdots\cdots+G\,N^{n+1}\frac{R^{n+1}}{(n+1)!}$$

$$+(n+1)\,G\,N^{n+1}\frac{R\,\rho^{n+1}}{(n+1)!}+(n+1)\,{}_nC_1\,G.\,N^{n+1}\frac{R^2\rho^n}{2!\,n!}$$

$$+(n+1)\,{}_nC_2\,G\,N^{n+1}\frac{R^3\rho^n}{3!\,(n-1)!}+\cdots\cdots+(n+1)\,G\,N^{n+1}\frac{\rho\,R^{n+1}}{(n+1)!}$$

$$+{}_{n+1}C_2\,G\,N^{n+1}\frac{R^2\rho^{n+1}}{2!\,(n+1)!}+{}_{n+1}C_2\,{}_{n-1}C_1\,G\,N^{n+1}\frac{R^3\rho^n}{3!\;n!}$$

$$+\cdots\cdots+G\,N^{n+1}\frac{R^{n+1}\rho^2}{2!\,(n+1)!}+\cdots\cdots+G\,N^{n+1}\frac{R^{n+1}\rho^{n+1}}{(n+1)!^2}$$

$$<G\,N^{n+1}\frac{(\rho+R)^{n+1}}{\left(I\left(\frac{n+1}{2}\right)!\right)^2}+G\,N^{n+1}\,{}_{n+1}C_1\,R\,\rho\frac{(R+\rho)^n}{\left(I\left(\frac{n+1}{2}\right)!\right)^2}$$

$$+G\,N^{n+1}\,{}_{n+1}C_2\,R^2\rho^2\frac{(R+\rho)^{n-1}}{\left(I\left(\frac{n+1}{2}\right)!\right)^2}+\cdots\cdots+G\,N^{n+1}\frac{R^{n+1}\,\rho^{n+1}}{\left(I\left(\frac{n+1}{2}\right)!\right)^2}$$

$$<G\,N^{n+1}\frac{(\rho+R+R\rho)^{n+1}}{\left(I\left(\frac{n+1}{2}\right)!\right)^2},$$

wobei $I\left(\frac{n+1}{2}\right)$ die grösste in $\frac{n+1}{2}$ enthaltene ganze Zahl bedeutet.

Da aber die Reihe

$$\sum G\,N^{n+1}\frac{(\rho+R+R\rho)^{n+1}}{\left(I\left(\frac{n+1}{2}\right)!\right)^2}$$

konvergiert, so konvergiert

$$\Phi_n(x,y)$$

gleichmässig gegen eine reguläre analytische Funktion $\Phi(x,y)$.

Aus der Relation

$$\Phi_{n+1}(x,y)=\varphi(x,y)-\int_0^y K(y,\tau)\,\Phi_n(x,\tau)\,d\tau$$

$$-\int_0^x L(x,t)\,\Phi_n(t,y)\,dt-\int_0^x\int_0^y M(x,t;\,y,\tau)\,\Phi_n(t,\tau)\,dt\,d\tau,$$

folgt

$$\Phi(x,y)=\varphi(x,y)-\int_0^x K(y,\tau)\,\Phi(x,\tau)\,d\tau-\int_0^x L(x,t)\,\Phi(t,y)\,dt$$
$$-\int_0^x\int_0^y M(x,t;\,y,\tau)\,\Phi(t,\tau)\,dt\,d\tau,$$

und $\Phi(x,y)$ ist die gesuchte Funktion.

Als Spezialfall der obigen Gleichung ist die Existenz der analytischen Lösung der Integralgleichung (2) bewiesen, woraus sich ergibt die Existenz der regulären analylischen Lösung der Gleichung

$$\psi(x,y)=\int_0^x\int_0^y P(x-t,\ y-\tau)\,\Phi(t,\tau)\,dt\,d\tau.$$

Schliesslich ist

$$f(x,y)=\frac{f(0,0)}{P(0,0)}\,P(x,y)$$
$$+\int_0^x P(x-t,y)\,F(t)\,dt+\int_0^y P(x,y-\tau)\,G(\tau)\,d\tau$$
$$+\int_0^x\int_0^y P(x-t,y-\tau)\,\Phi(t,x)\,dt\,d\tau$$

und folglich

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}P_{mn}(x,y).$$

Damit ist der vorgelegte Satz bewiesen.

Den 29 ten Mai, 1922.

Grundlagen einer allgemeinen Theorie der zeitlich veränderlichen Vektorfelder und ihrer Relativitätstheorie,

von

LUCIUS HANNI in Wien.

In einer früheren Arbeit des Verfassers(1) wurde gezeigt, wie man mittels funktionentheoretischer Betrachtungen durch Übertragung der Cauchy-Riemannschen Gleichungen auf Biquaternionen $\mathfrak{U}+\mathfrak{V}_i$, die von vier reellen Veränderlichen x_1, x_2, x_3, t abhängen, zu den beiden Systemen von Gleichungen gelangt

$$
1\,a)\quad
\begin{aligned}
&\frac{\partial u_4}{\partial x_1}-\frac{\partial v_1}{\partial t}+\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}=0\\
&\frac{\partial u_4}{\partial x_2}-\frac{\partial v_2}{\partial t}+\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}=0\\
&\frac{\partial u_4}{\partial x_3}-\frac{\partial v_3}{\partial t}+\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}=0\\
&\frac{\partial v_4}{\partial t}+\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0
\end{aligned}
\qquad
1\,b)\quad
\begin{aligned}
&\frac{\partial v_4}{\partial x_1}+\frac{\partial u_1}{\partial t}+\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}=0\\
&\frac{\partial v_4}{\partial x_2}+\frac{\partial u_2}{\partial t}+\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}=0\\
&\frac{\partial v_4}{\partial x_3}+\frac{\partial u_3}{\partial t}+\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}=0\\
&\frac{\partial u_4}{\partial t}-\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}-\frac{\partial v_3}{\partial x_3}=0,
\end{aligned}
$$

wenn

$$\mathfrak{U}=u_1 j_1+u_2 j_2+u_3 j_3+u_4,$$

$$\mathfrak{V}=v_1 j_1+v_2 j_2+v_3 j_3+v_4$$

ist und diese Funktionen innerhalb des betreffenden Bereiches den entsprechenden Stetigkeits- und Differenzierbarkeitsbedingungen genügen. Da aus diesen beiden Gleichungssystemen folgt, dass $\mathfrak{U}$ und $\mathfrak{V}$ der Wellengleichung genügen, so kann man sich umgekehrt die Aufgabe stellen, von Wellen im dreidimensionalen Raum auszugehen und sodann die Gleichungen 1 a, b) einzuführen. Für den speziellen Fall, wo u_4 und v_4 verschwinden und die Gleichungen 1 a, b) in die Maxwellschen Gleichungen

(1) Ü. den Zusammenhang zwischen den Cauchy-Riemannschen u. den Maxwellschen Differentialgleichungen. Dieses Journal, vol 5, 1914.

für homogene isotrope Nichtleiter übergehen, ist diese Aufgabe schon vom Verfasser gelöst[1].

Hier wollen wir nun die Gleichungen 1 a, b) selbst einführen, indem wir von Wellenbewegungen im dreidimenisionalen Raum ausgehen, die transversale und longitudinale Wellen zugleich enthalten. Von da können wir unmittelbar zu beliebigen zeitlich veränderlichen Vektorfeldern übergehen und erhalten eine Verallgemeinerung der Gleichungen 1a, b), an die wir den Energie-Impulssatz für beliebige zeitlich veränderliche Vektorfelder anschliessen können. Man gelangt so ohne Verwendung von Sätzen der Physik zu einer Theorie der Vektorfelder, die Gleichungen von derselben Form wie die Grundgleichungen der Elektrodynamik und der Mechanik enthält und ausserdem noch die Möglichkeit einer physikalischen Theorie der Gravitation im Anschluss an die Einsteinsche Gravitationstheorie bietet, da man einem zeitlich veränderlichen Vektorfeld im dreidimensionalen Raume umkehrbar eindeutig einen vierdimensionalen Raum zuordnen kann, für den die Riemann-Weylsche Geometrie gilt. Zum Schlusse zeigen wir noch, dass die Gleichungen 1 a, b) und der daraus sich ergebende Energie-Impulssatz von der Wahl eines Koordinatensystems im vierdimensionalen Raume unabhängig sind.

§ 1. Einführung der Wellengleichung und Kriterien für Longitudinalität und Transversalität von Wellen.

Wir betrachten zunächst folgenden einfachen Fall. Ein Vektor $\mathfrak{s}$ im dreidimensionalen Raume sei eine periodische Funktion der unabhängigen reellen Veränderlichen r und t, so dass sich $\mathfrak{s}$ in der Form darstellen lässt

$$\mathfrak{s}=\mathfrak{f}\,(r+t)+\mathfrak{g}\,(r-t),$$

wo die Vektoren $\mathfrak{f}$ und $\mathfrak{g}$ beliebig vorgegebene Funktionen sind, die innerhalb eines gegebenen Bereiches bestimmten Stetigkeits- und Differenzierbarkeitsbedingungen genügen sollen; ausserdem setzen wir fest, dass z einen Abstand von einem gegebenen Punkte und t die von einem gewissen Zeitpunkte an gemessene Zeit bedeute. Wir legen nun ein rechtwinkliges Koordinatensystem zugrunde und tragen die dem vorgegebenen Bereiche angehörenden Werte von r auf einer Geraden durch den Ursprung des Koordinatensystems auf, deren Richtungscosinus α, β, γ sein mögen. Sind ξ, η, ζ die drei Projektionen von $\mathfrak{s}$, so bestehen also die drei Wellengleichungen

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[1] Einführung der Maxwellschen Gleichungen in die Wellenlehre. Dieses Journal, vol. 20, 1921.

$$2)\qquad \frac{\partial^2\xi}{\partial t^2}=\frac{\partial^2\xi}{\partial r^2},\ \frac{\partial^2\eta}{\partial t^2}=\frac{\partial^2\eta}{\partial r^2},\ \frac{\partial^2\zeta}{\partial t^2}=\frac{\partial^2\zeta}{\partial r^2}.$$

Diese Wellenbewegung kann man sich nun dadurch in der üblichen Weise veranschaulichen, dass man den Vektor $\mathfrak{s}$ in zwei Komponenten $\mathfrak{s}'$ und $\mathfrak{s}''$ zerlegt, in eine parallel zur Fortpflanzungsrichtung der Wellenbewegung mit den Richtungscosinus α, β, γ und in eine zur Fortpflanzungsrichtung senkrechte. Die zur Fortpflanzungsrichtung parallele Komponente $\mathfrak{s}'$ von $\mathfrak{s}$ genügt der Bedingungsgleichung

$$3)\qquad (\zeta'\beta-\eta'\gamma)^2+(\xi'\gamma-\zeta'\alpha)^2+(\eta'\alpha-\xi'\beta)^2=0$$

und die zur Fortpflanzungsrichtung senkrechte Komponente $\mathfrak{s}''$ von $\mathfrak{s}$ erhält man durch die Bedingung

$$4)\qquad \xi''\alpha+\eta''\beta+\zeta''\gamma=0.$$

Im ersten Falle stellen die Gleichungen 2) longitudinale Wellen dar, im zweiten Falle transversale Wellen, während im allgemeinen Falle transversale und longitudinale Wellen zugleich bestehen.

Man kann sich jetzt die Aufgabe stellen, die Wellengleichungen 2) so umzuformen, dass man aus der Form der Wellengleichung selbst sieht, welche Art von Wellen gegeben seien. Dazu ist notwendig, dass man die Richtungscosinus α, β, γ in die Wellengleichung einführt, so dass neben r auch noch die Richtungscosinus α, β, γ darin auftreten. Anstatt der Grössen r, α, β, γ kann man aber auch die rechtwinkligen Koordinaten x, y, z der Punkte auf der Fortpflanzungsrichtung verwenden, die der Bedingung

$$5)\qquad x=r\alpha,\ y=r\beta,\ z=r\gamma$$

genügen, so dass

$$r=x\alpha+y\beta+z\gamma$$

ist. Es ist dann

$$\frac{\partial r}{\partial x}=\alpha,\ \frac{\partial r}{\partial y}=\beta,\ \frac{\partial r}{\partial z}=\gamma.$$

Um jetzt z. B. in die erste der Gleichungen 2) die Veränderlichen x, y, z einführen zu können, schreiben wir sie in der Form

$$\frac{\partial^2\xi}{\partial t^2}=\frac{\partial^2\xi}{\partial r^2}(\alpha^2+\beta^2+\gamma^2).$$

Mit Rücksicht auf Gleichungen von der Form

$$\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial \xi}{\partial r} \alpha$$

und

$$\frac{\partial^2 \xi}{\partial x^2} = \alpha \frac{\partial \frac{\partial \xi}{\partial r}}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial^2 \xi}{\partial r^2} \cdot \alpha^2$$

geht die erste der Gleichungen 2) in die Gleichung

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}$$

über. Man erhält so an Stelle der Gleichungen 2) das Gleichungssystem

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2},$$

6)
$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2},$$

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}.$$

Die Gleichungen 6) haben jetzt schon eine solche Gestalt, dass man die Ausdrücke 3) und 4) in dieselben einführen kann, nachdem man 3) und 4) nach r differenziert hat und sodann an Stelle von α, β, γ die Koordinaten x, y, z verwendet. Es sagt dann die Bedingung 3) aus, dass der Vektor $\mathfrak{u}$ mit den Komponenten

$$u_x = \frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}, \quad u_y = \frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}, \quad u_z = \frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}$$

verschwinden soll, während die Bedingung 4) in die Gleichung

7)
$$u_4 = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0$$

übergeht. Mit Rücksicht auf Identitäten von der Form

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)$$
$$+ \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right)$$

kann man nun die Gleichungen 6) in der Form schreiben

$$
6') \qquad \begin{aligned}
\frac{\partial^2 \xi}{\partial t^2} &= \frac{\partial u_4}{\partial x} - \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial x}\right), \\
\frac{\partial^2 \eta}{\partial t^2} &= \frac{\partial u_4}{\partial y} - \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \\
\frac{\partial^2 \zeta}{\partial t^2} &= \frac{\partial u_4}{\partial z} - \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right).
\end{aligned}
$$

Hiemit haben wir die Wellengleichungen 2) schon auf die gewünschte Form gebracht.

Die Bedeutung der Ausdrücke $\mathfrak{u}$ und u_4 lässt sich leicht angeben. Die Bedeutung des Vektors $\mathfrak{u}$ ist bei transversalen Wellen, die sich längs einer einzelnen Geraden fortpflanzen, schon in der zweiten der oben citierten Arbeiten des Verfassers angegeben, während die Bedeutung von u_4 in dem hier betrachteten Falle als Maass für eine lineare Verdichtung bzgw. Verdünnung im Gegensatz zur Volumdilatation schon bekannt ist. Aus der Bedeutung von $\mathfrak{u}$ und u_4 ergibt sich umgekehrt wieder leicht, dass ihr Verschwinden eine Bedingung für die Existenz von longitudinalen Wellen allein bzgw. von transversalen Wellen allein darstellt.

§ 2. Longitudinale Wellen.

Nach diesen Bemerkungen gelangen wir schon zu jenem speziellen Fall der Gleichungssysteme 1 a, b), den man erhält, wenn man voraussetzt, dass nur longitudinale Wellen gegeben seien, die sich längs einer einzelnen Geraden fortpflanzen. Wir nehmen also an, es genügen die Funktionen ξ', η', ζ' den Gleichungen 6) und es bestehen neben den Gleichungen 5) die Gleichungen

$$
8) \qquad \frac{\partial \zeta'}{\partial y} - \frac{\partial \eta'}{\partial z} = 0, \quad \frac{\partial \xi'}{\partial z} - \frac{\partial \zeta'}{\partial x} = 0, \quad \frac{\partial \eta'}{\partial x} - \frac{\partial \xi'}{\partial y} = 0.
$$

Aus diesen letzten Gleichungen ergibt sich, dass

$$
9) \qquad \xi' = \frac{\partial \phi}{\partial x}, \quad \eta' = \frac{\partial \phi}{\partial y}, \quad \zeta' = \frac{\partial \phi}{\partial z}
$$

ist, wo ϕ eine Potentialfunktion ist. Setzt man

$$
\frac{\partial \phi}{\partial t} = \varphi_4',
$$

so bestehen also zufolge der Gleichungen 9) auch die Gleichungen

10) $$\frac{\partial \xi'}{\partial t}=\frac{\partial \varphi_4'}{\partial x},\quad \frac{\partial \eta'}{\partial t}=\frac{\partial \varphi_4'}{\partial y},\quad \frac{\partial \zeta'}{\partial t}=\frac{\partial \varphi_4'}{\partial z}.$$

Bevor wir nun dem den longitudinalen Wellen entsprechenden speziellen Fall der Gleichungssysteme 1*a*, *b*) einführen können, müssen wir von den Gleichungen 6) für longitudinale Wellen zu einem solchen System von Wellengleichungen übergehen, in welchem zu dem von x, y, z und t abhängigen Vektor $\mathfrak{s}'$ im dreidimensionalen Raume noch eine von x, y, z und t abhängige skalare Funktion φ_4' hinzutritt. Um zu einem solchen Gleichungssystem zu gelangen, führen wir zunächst mittels der Beziehungen 8) und 10) die Gleichungen 6) auf eine einfachere Form zurück. Dabei wollen wir schon jetzt die in den Gleichungen 1a, b) verwendete Bezeichnungsweise einführen, indem wir die unabhängigen Veränderlichen x, y, z mit x_1, x_2, x_3 und die abhängigen Veränderlichen ξ', η', ζ' mit φ_1', φ_2', φ_3' bezeichnen. Wir betrachten nun die erste der Gleichungen 6). Zufolge 8) und 10) kann man in Übereinstimmung mit den Gleichungen 6′) diese Gleichung in der Form schreiben

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi_4'}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\left(\frac{\partial \varphi_1'}{\partial x_1}+\frac{\partial \varphi_2'}{\partial x_2}+\frac{\partial \varphi_3'}{\partial x_3}\right).$$

Behandelt man die zweite und dritte der Gleichungen 6) in analoger Weise, so nehmen die Gleichungen 6) jetzt die Form an

11*a*) $$\frac{\partial}{\partial x_i}\left(\frac{\partial \varphi_1'}{\partial x_1}+\frac{\partial \varphi_2'}{\partial x_2}+\frac{\partial \varphi_3'}{\partial x_3}-\frac{\partial \varphi_4'}{\partial t}\right)=0,\ (i=1,\,2,\,3),$$

wenn man voraussetzt, dass die Reihenfolge der Differentiation vertauscht werden darf.

Von hier gelangt man jetzt leicht dazu, für longitudinale Wellen neben die drei Gleichungen 6) noch eine vierte analoge Gleichung für den Skalar φ' aufzustellen. Zufolge der Gleichungen 11a) kann nämlich der Ausdruck in der Klammer höchstens noch eine Funktion $\psi(t)$ von t sein. Diese Tatsache kann man auch in der folgenden Form darstellen

$$\frac{\partial \varphi'_1}{\partial x_1}+\frac{\partial \varphi_2'}{\partial x_2}+\frac{\partial \varphi_3'}{\partial x_3}-\frac{\partial \{\varphi_4'+\varphi(t)\}}{\partial t}=0,$$

wo

$$\frac{\partial \varphi(t)}{\partial t}=\psi(t)$$

ist. Die Gleichungen 10) bleiben aber noch bestehen, wenn man darin φ_4' durch $\varphi_4'+\varphi(t)$ ersetzt. Bezeichnet man $\varphi_4+\varphi(t)$ wieder mit φ_4, so bestehen also für longitudinale Wellen neben 5) die Gleichungen 8) und 10) und die Gleichungen 6) können durch die Gleichung

$$11) \qquad \frac{\partial\varphi_1'}{\partial x_1}+\frac{\partial\varphi_2'}{\partial x_2}+\frac{\partial\varphi_3'}{\partial x_s}-\frac{\partial\varphi_4'}{\partial t}=0$$

ersetzt werden. Differenziert man die Gleichung 11) nach t, so folgt wegen des Bestehens der Gleichungen 10), dass auch φ_4' der Gleichung

$$\frac{\partial^2\varphi'_4}{\partial t^2}=\frac{\partial^2\varphi_4'}{\partial x_1^2}+\frac{\partial^2\varphi_4'}{\partial x_2^2}+\frac{\partial^2\varphi_4'}{\partial x_3^2}$$

genügt. Es lässt sich so für longitudinale Wellen das ursprünglich gegebene System 6) durch das Gleichungssystem

$$12) \qquad \frac{\partial^2\varphi_i'}{\partial t^2}=\frac{\partial^2\varphi_i'}{\partial x_1^2}+\frac{\partial^2\varphi_i'}{\partial x_2^2}+\frac{\partial^2\varphi_i'}{\partial x_3^2},\ (i=1,2,3,4)$$

ersetzen. Man kann also bei longitudinalen Wellen schon durch Einführung der skalaren Funktion φ_4' von den Gleichungen 6) des dreidimensionalen Raumes zu den Gleichungen 12) des vierdimensionalen Raumes übergehen, ohne irgend eine neue Voraussetzung machen zu müssen, und man sieht zugleich, dass für longitudinale Wellen das Gleichungssystem 12) nicht mehr aussagt als das System 6).

Aus diesen Überlegungen ergibt sich, dass es bei der Charakterisierung des Verlaufes von longitudinalen Wellen, die längs einer Geraden mit den Richtungscosinus α, β, γ fortschreiten, auf den Ausdruck

$$\frac{\partial(\varphi_1'\alpha+\varphi_2'\beta+\varphi_3'\gamma)}{\partial z}=\frac{\partial\varphi_1'}{\partial x_1}+\frac{\partial\varphi_2'}{\partial x_2}+\frac{\partial\varphi_3'}{\partial x_3}=u_4$$

und auf die skalare Funktion φ_4 ankommt. Im Anschluss an die Gleichungen 12) ergeben sich nun bei longitudinalen Wellen sehr einfache Beziehungen zwischen den Komponenten des Gradienten

$$u_1'=-\frac{\partial\varphi_4'}{\partial x_1},\ u_2'=-\frac{\partial\varphi_4'}{\partial x_2},\ u_3'=-\frac{\partial\varphi_4'}{\partial x_3},$$

den Differentialquotienten von u_4 und den Differentialquotienten

$$v_1'=\frac{\partial\varphi_1'}{\partial t},\ v_2'=\frac{\partial\varphi_2'}{\partial t},\ v_3'=\frac{\partial\varphi_3'}{\partial t},\ v_4'=\frac{\partial\varphi_4'}{\partial t};$$

man gelangt dann gerade zu dem für longitudinale Wellen geltenden speziellen Fall der Gleichungen 1a, b). Aus den Gleichungen 12) erhält man nämlich wegen des Bestehens der Gleichungen 8) das Gleichungssystem

13a)
$$\begin{gathered}
\frac{\partial u_4}{\partial x_1}-\frac{\partial v_1'}{\partial t}=0,\\
\frac{\partial u_4}{\partial x_2}-\frac{\partial v_2'}{\partial t}=0,\\
\frac{\partial u_4}{\partial x_3}-\frac{\partial v_3'}{\partial t}=0,\\
\frac{\partial v_4}{\partial t}+\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_2'}{\partial x_2}+\frac{\partial u_3'}{\partial x_3}=0\,;
\end{gathered}$$

ausserdem folgt aus der Definition des durch φ_4' gegebenen Gradienten, dass seine Komponenten u_1', u_2', u_3' den Gleichungen genügen.

13b) $$\frac{\partial u_3'}{\partial x_2}-\frac{\partial u_2'}{\partial x_3}=0,\qquad \frac{\partial u_1'}{\partial x_3}-\frac{\partial u_3'}{\partial x_1}=0,\qquad \frac{\partial u_2'}{\partial x_1}-\frac{\partial u_1'}{\partial x_2}=0.$$

Für longitudinale Wellen geht also das Gleichungssystem 1 *a*) in das System 13 *a*) über, zu dem noch das aus den Definitionsgleichungen von u_1', u_2', u_3' sich ergebende System 13 *b*) hinzutritt.

Um auch zum System 1 *b*) zu gelangen, wollen wir davon ausgehen, dass in den Systemen 13 *a*, *b*) gerade 16 der 32 Differentialquotienten von u_1', u_2', u_3', u_4 und v_1', v_2', v_3', v_4 auftreten; es handelt sich also noch darum, Beziehungen zwischen den übrigen 16 Differentialquotienten zu finden. Diese Beziehungen ergeben sich aus den Definitionsgleichungen der Funktionen u_1', u_2', u_3', u_4 und v_1', v_2', v_3', v_4 und den Bedingungen 8). Aus den Definitionsgleichungen von v_4 und u_1', u_2', u_3', u_4 folgen die Gleichungen

14 *a*)
$$\begin{gathered}
\frac{\partial v_4}{\partial x_1}+\frac{\partial u_1'}{\partial t}=0\\
\frac{\partial v_4}{\partial x_2}+\frac{\partial u_2'}{\partial t}=0\\
\frac{\partial v_4}{\partial x_3}+\frac{\partial u_3'}{\partial t}=0\\
\frac{\partial u_4}{\partial t}-\frac{\partial v_1'}{\partial x_1}-\frac{\partial v_2'}{\partial x_2}-\frac{\partial v_3'}{\partial x_3}=0
\end{gathered}$$

und aus den Definitionsgleichungen von v_1', v_2', v_3' und den Bedingungsgleichungen 8) folgen die Gleichungen

14.b) $$\frac{\partial v_3'}{\partial x_2} - \frac{\partial v_2'}{\partial x_3} = 0, \quad \frac{\partial v_1'}{\partial x_3} - \frac{\partial v_3'}{\partial x_1} = 0, \quad \frac{\partial v_2'}{\partial x_1} - \frac{\partial v_1'}{\partial x_2} = 0.$$

Bei dieser Einführung des für longitudinale Wellen sich ergebenden speziellen Falles der Gleichungen 1a, b) ist noch immer vorausgesetzt, dass die Bedingungsgleichungen 5) bestehen. Lässt man diese Gleichungen weg, so sind durch die Gleichungen 6) oder 12) zusammen mit den Gleichungen 8) longitudinale Wellen in einem homogenen isotropen Medium gegeben. Den Verlauf von longitudinalen Wellen in einem homogenen isotropen Medium kann man nun wieder in derselben Weise untersuchen wie im III. Teile der zweiten der oben citierten Arbeiten des Verfassers den Verlauf von transversalen Wellen. Es ergibt sich dann, dass auch die Fortpflanzungsrichtungen von longitudinalen Wellen in einem homogenen isotropen Medium eine Strahlenkongruenz bilden. Von den Wellengleichungen 12) zusammen mit den Gleichungen 8) und 10) ohne die Bedingungsgleichungen 5) kann man jetzt wieder in derselben Weise wie oben zu den Gleichungen 13a, b) und 14a, b) übergehen, da ihre Einführung von den Bedingungsgleichungen 5) unabhängig ist.

Nachdem für longitudinale Wellen die entsprechende spezielle Form der Gleichungen 1a, b) eingeführt ist, wollen wir noch die so erhaltenen Gleichungen 13a) in einer andern mehr symmetrischen Form schreiben und zugleich zeigen, wie sich auch diese Form unmittelbar aus den Wellengleichungen 12) ergibt. Wir gehen also dabei von den Gleichungen 12) aus und beschränken sie nicht mehr auf longitudinale Wellen allein, was wir dadurch andeuten, dass wir darin φ_i' durch φ_i ersetzen. Setzt man sodann in 12)

$$ti = x_4, \quad (i = \sqrt{-1}),$$

so muss auch φ_4 durch $\varphi_4 i$ ersetzt werden; doch führen wir aus Symmetriegründen an Stelle von φ_4 die Funktion $-\varphi_4 i$ ein. Setzen wir noch

$$a_{ik} = \frac{\partial \varphi_i}{\partial x_k}, \quad (i, k = 1, 2, 3, 4)$$

so nehmen die Gleichungen 12) die Form an

12') $$\sum_{k=1}^{4}{}' \frac{\partial a_{ik}}{\partial x_k} = 0, \quad (i = 1, 2, 3, 4).$$

Wir setzen nun

$$g_{ik}=a_{ik}+a_{ki}=\frac{\partial\varphi_i}{\partial x_k}+\frac{\partial\varphi_k}{\partial x_i}$$

$$(i,\ k=1,\ 2,\ 3,\ 4)$$

$$f_{ik}=a_{ik}-a_{ki}=\frac{\partial\varphi_i}{\partial x_k}-\frac{\partial\varphi_k}{\partial x_i},$$

so dass

$$g_{ik}=g_{ki},\quad f_{ik}=-f_{ki},\quad f_{ii}=0$$

ist. Es sind dann die Funktionen g_{k4} und f_{k4}, $(k=1,\ 2,\ 3)$, rein imaginäre Grössen. Denn es ist

$$g_{k4}=\frac{\partial\varphi_k}{\partial t_i}-\frac{i\partial\varphi_4}{\partial x_k}=-i\left(\frac{\partial\varphi_k}{\partial t}+\frac{\partial\varphi_4}{\partial x_k}\right),$$

wo φ_4 wieder die ursprünglich gegebene reelle skalare Funktion ist; ebenso ist

$$f_{k4}=\frac{\partial\varphi_k}{\partial ti}+\frac{i\partial\varphi_4}{\partial x_k}=-i\left(\frac{\partial\varphi_k}{\partial t}-\frac{\partial\varphi_4}{\partial x_k}\right).$$

Da bei longitudinalen Wellen wegen des Bestehens der Gleichungen 8) und 10) sämtliche f_{ik} verschwinden, so nehmen bei longitudinalen Wellen die durch obige Transformation aus den Wellengleichungen 12) sich ergebenden Gleichungen 12′) die Form an

13a') $$\sum_{k=1}\frac{\partial g_{ik}}{\partial x_k}=0,\quad (i=1,\ 2,\ 3,\ 4),$$

zu denen wegen 11) noch die Gleichung

11′) $$g_{11}+g_{22}+g_{33}+g_{44}=0$$

hinzutritt. Bei longitudinalen Wellen geht nun das Gleichungssystem 13a') gerade durch die hier angegebene Transformation der vierten Koordinate aus dem System 13a) hervor, so dass dann die Gleichungssysteme 13a) und 13a') einander äquivalent sind.

Ist umgekehrt ein Gleichungssystem von der Form 13a') gegeben, so kann man von ihm wieder zu den entsprechenden Gleichungen für longitudinale Wellen zurückgehen, wenn die Gleichung 11′) besteht. Mittels der gegebenen Tensorkomponenten g_{ik} kann man nämlich schon Funktionen φ_i definieren, die den Differentialgleichungen genügen

$$d\varphi_i=\frac{1}{2}\sum_{k=1}^{4}g_{ik}\,dx_k,\quad (i=1,\ 2,\ 3,\ 4).$$

Die so definierten Funktionen φ_i haben dann schon die Eigenschaft, dass die Funktionen g_{ik} den oben angegebenen Definitionsgleichungen genügen und sämtliche Funktionen f_{ik} verschwinden; ausserdem geht wegen des Bestehens von 11') jede der Gleichungen 13a') in die betreffende aus der Wellengleichung sich ergebende Laplacesche Gleichung über.

Ebenso wie man unter der Voraussetzung, dass die Funktionen f_{ik} verschwinden, die Gleichungen 12) und 13a) in Beziehungen zwischen den Funktionen g_{ik} verwandeln kann, so kann man unter derselben Voraussetzung auch die Gleichungen 13b) u.s.w. in Beziehungen zwischen den Funktionen g_{ik} verwandeln.

Die Gleichungen 13a) und 13a') unterscheiden sich dadurch von einander, dass zu ihrer Darstellung das einemal wenigstens implizit Quaternionen, das anderemal die Tensorrechnung verwendet wurden; natürlich liessen sich schon hier Beispiele für die Nebeneinanderstellung beider Rechnungsarten beliebig vermehren. Um von der einen Darstellung zur anderen leicht übergehen zu können, wurden daher hier die abkürzenden Bezeichnungen der Analysis der Quaternionen nicht verwendet.

Endlich ist über die hier gewählte Bezeichnung der unabhängigen Veränderlichen zu bemerken, dass bei Wellen im dreidimensionalen Raume zunächst die Bezeichnung x, y, z, t gewählt wurde, nach Einführung der Funktion φ_4 die Bezeichnung x_1, x_2, x_3, t verwendet wurde, so dass erst nach dem Übergang von der Wellengleichung zur Laplaceschen Gleichung die ganz symmetrische Bezeichnung x_1, x_2, x_3, x_4 angewendet wird; für die Verwendung der Bezeichnung x_1, x_2, x_3, t war ausserdem noch massgebend, dass für die Existenz von Wellenbewegungen und ihre Charakterisierung als longitudinale und transversale Wellen nur der Ausdruck auf der rechten Seite der Wellengleichung in Betracht kommt.

§ 3. Transversale Wellen.

Wir nehmen jetzt an, es seien transversale Wellen allein gegeben und ξ'', η'', ζ'' seien die Komponenten des Vektors $\mathfrak{s}''$, ferner sei dementsprechend

$$v_x'' = \frac{\partial \xi''}{\partial t}, \quad v_y'' = \frac{\partial \eta''}{\partial t}, \quad v_z'' = \frac{\partial \zeta''}{\partial t},$$

endlich sollen ξ'', η'', ζ'' der Gleichung 7) genügen, so dass

$$\frac{\partial \xi''}{\partial x} + \frac{\partial \eta''}{\partial y} + \frac{\partial \zeta''}{\partial z} = 0$$

ist, und es sei

$$u_x'' = \frac{\partial \zeta''}{\partial y} - \frac{\partial \eta''}{\partial z}, \; u_y'' = \frac{\partial \xi''}{\partial z} - \frac{\partial \zeta''}{\partial x}, \; u_z'' = \frac{\partial \eta''}{\partial x} - \frac{\partial \xi''}{\partial y}.$$

In analoger Weise wie bei longitudinalen Wellen kann man auch den für transversale Wellen sich ergebenden speziellen Fall der Gleichungssysteme 10, b) einführen, indem man davon ausgeht, dass für ξ'', η'', ζ'' die Wellengleichung 6) und die Gleichung 7) bestehen. Wie in der zweiten der oben citierten Arbeiten des Verfassers bereits durchgeführt wurde, erhält man dann als speziellen Fall der Gleichungen 1 a, b) die beiden Systeme von Maxwellschen Gleichungen

$$15\,a)\quad \begin{aligned} \frac{\partial v_x''}{\partial t} &= \frac{\partial u_z''}{\partial t} - \frac{\partial u_y''}{\partial z} \\ \frac{\partial v_y''}{\partial t} &= \frac{\partial u_x''}{\partial z} - \frac{\partial u_z''}{\partial x} \\ \frac{\partial v_z''}{\partial t} &= \frac{\partial u_y''}{\partial x} - \frac{\partial u_x''}{\partial y} \\ \frac{\partial u_x''}{\partial x} &+ \frac{\partial u_y''}{\partial y} + \frac{\partial u_z''}{\partial z} = 0, \end{aligned} \qquad 15\,b)\quad \begin{aligned} \frac{\partial u_x''}{\partial t} &= \frac{\partial v_y''}{\partial z} - \frac{\partial v_z''}{\partial y} \\ \frac{\partial u_y''}{\partial t} &= \frac{\partial v_z''}{\partial x} - \frac{\partial v_x''}{\partial z} \\ \frac{\partial u_z''}{\partial t} &= \frac{\partial v_x''}{\partial y} - \frac{\partial v_y''}{\partial z} \\ \frac{\partial v_x''}{\partial x} &+ \frac{\partial v_y''}{\partial y} + \frac{\partial v_z''}{\partial z} = 0. \end{aligned}$$

Nachdem im Anschluss an die Gleichungen 6) und 7) die Gleichungen 57 a, b) eingeführt sind, kann man auch bei transversalen Wellen von den Wellengleichungen 6) für den dreidimensionalen Raum zu Wellengleichungen von der Form 12) für den vierdimensionalen Raum übergehen, indem man noch eine Potentialfunktion φ_4'' einführt. Dabei können wir davon ausgehen, dass die Gleichungen 15 a, b) noch bestehen bleiben, wenn man an Stelle des Vektors mit den Komponenten v_x'', v_y'', v_z'' den Vektor mit den Komponenten

$$v_x''' = \frac{\partial \xi''}{\partial t} - \frac{\partial \varphi_4''}{\partial x}, \; v_y''' = \frac{\partial \eta''}{\partial t} - \frac{\partial \varphi_4''}{\partial y}, \; v_z''' = \frac{\partial \zeta''}{\partial t} - \frac{\partial \varphi_4''}{\partial z}$$

einführt, falls $\frac{\partial \varphi_4''}{\partial t}$ verschwindet und $\varphi_4{}'$ der Gleichung

$$\frac{\partial^2 \varphi_4''}{\partial x^2} + \frac{\partial^2 \varphi_4''}{\partial y^2} + \frac{\partial^2 \varphi_4''}{\partial z^2} = 0$$

genügt; diese letztere Bedingung ist notwendig, wenn die letzte der Gleichungen 15 b) bestehen soll. Fügt man diese letzte Gleichung zu den Wellengleichungen 6) für die Komponenten ξ'', η'', ζ'', so erhält man auch für transversale Wellen ein Gleichungssystem von der Form 12), bei dem allerdings in der letzten Gleichung das Glied $\frac{\partial^2 \varphi_4''}{\partial t^2}$ wegfällt, da auch die neuen Funktionen v_x''', v_y''', v_z'' der Gleichung 7) genügen sollen.

Wenn auch durch die hier angegebene Einführung der Potentialfunktion φ_4'' die Gleichungen 15 a, b) noch bestehen bleiben und die Vektor mit den Komponenten v_x''', v_y''', v_z''' in derselben Weise interpretiert werden kann wie in der zweiten der oben citierten Arbeiten des Verfassers der Vektor mit den Komponenten v_x'', v_y'', v_z'', so sind diese beiden transversalen Wellenbewegungen doch nicht identisch. Betrachtet man nämlich transversale Wellen, die längs einer einzelnen Geraden fortschreiten, so dass die Gleichungen 5) bestehen, und bildet man die Gleichung 7) anstatt für den Vektor mit den Komponenten v_x'', v_y'', v_z'' jetzt für den Vektor mit den Komponenten v_x''', v_y''', v_z''', so sieht man, dass dann die Fortpflanzungsrichtungen der beiden Wellenbewegungen verschieden sind; man braucht ja nur von der Gleichung 7) zur Gleichung 4) zurückzugehen und zu beachten, dass jetzt die Komponenten v_x''', v_y''', v_z''' vorgegeben sind und daher die Richtungscosinus α, β, γ andere Werte annehmen müssen. Man erhält so schon durch Betrachtungen aus der Theorie der Vektorfelder das interessante Resultat, dass transversale Wellen durch ein Potentialfeld abgelenkt werden.

Nachdem man in die Gleichungen 15 a, b) die Komponenten v_x''', v_y''', v_z''' eingeführt hat, kann man jetzt leicht in diese Gleichungen die bereits im früheren Paragraphen definierten Funktionen f_{ik} einführen, indem man wieder x, y, z mit x_1, x_2, x_3 und ξ'', η'', ζ'' mit $\varphi_1'', \varphi_2'', \varphi_3''$ bezeichnet und ausserdem t durch $x_4 i$ und φ_4'' durch $-\varphi_4'' i$ ersetzt. Man erhält dann aus den Gleichungen 15 a, b) die Minkowskische symmetrische Form der Maxwellschen Gleichungen, wenn man die 4. Gleichung des Systems 15 a) mit der 4. des Systems 15 b) vertauscht. Nach Vertauschung dieser beiden Gleichungen nimmt insbesondere das System 15 a) die bekannte Form an

$$15\,a') \qquad \sum_{k=1}^{4} \frac{\partial f_{ik}}{\partial x_k} = 0, \quad (i = 1, 2, 3, 4).$$

Das Gleichungssystem 15 a') ergibt sich aber durch die hier verwendete Transformation auch unmittelbar aus den Gleichungen 12) und 7);

man kann nämlich die ersten 3 Gleichungen des Systems 12) in der Form 6′) schreiben und hat nur zu beachten, dass jetzt u_4 verschwindet, während die 4. der Gleichungen 15 a') aus der Gleichung 7) und der 4. Gleichung von 12) hervorgeht, welche letztere bei transversalen Wellen aussagt, dass φ_4'' eine Potentialfunktion im dreidimensionalen Raume ist. Man sieht so unmittelbar, wie die Minkowskische symmetrische Schreibweise der Maxwellschen Gleichungen sich an die von Wellengleichungen von der Form 12′) anschliesst.

Die Maxwellschen Gleichungen 15 a, b) und die entsprechende symmetrische Minkowskische Form dieser Gleichungen sind wieder ein Beispiel dafür, wie im vierdimensionalen Baume die Vektoranalysis und die Tensorrechnung neben einander verwendet werden können.

§ 4. Zusammensetzung von Wellenbewegungen.

Nach diesen Betrachtungen über longitudinale und transversale Wellen können wir jetzt zu dem Falle übergehen, wo in homogenen isotropen Medien longitudinale und transversale Wellen zugleich bestehen und gelangen so zu den Gleichungen 1 a, b) selbst. Dieser Übergang ist nur ein specieller Fall der Zusammensetzung von Wellenbewegungen überhaupt. Wir gehen dabei davon aus, dass die Komponenten ξ', η', ζ' und ξ'', η'', ζ'' von zwei Vektoren $\mathfrak{s}'$ und $\mathfrak{s}''$, die denselben Anfangspunkt haben, den Wellengleichungen 6) genügen. Diese Vektoren kann man nach den Gesetzen der Addition von Vektoren zusammensetzen und erhält so eine neue Wellenbewegung. Ist nun die erste von zwei Wellenbewegungen eine longitudinale und die zweite eine transversale, so kann man in jedem dieser beiden Fälle von den Gleichungen 6) auch zu den Gleichungen 12) übergehen, indem man die Funktion φ_4' und φ_4'' einführt. Bildet man so dann die Summe von $\mathfrak{s}'$ und $\mathfrak{s}''$ und die Summe

$$\varphi_4' + \varphi_4'' = \varphi_4,$$

so erhält man wieder ein Gleichungssystem von der Form 12), in dem jetzt die Summe φ_4 eine beliebig vorgegebene Funktion von x_1, x_2, x_3 und t ist, die nur noch der Wellengleichung genügen muss, wie sich aus der hier angegebenen Einführung von φ_4 und φ_4'' unmittelbar ergibt.

Durch Verwendung dieser einfachen Bemerkungen über die Zusammensetzung von longitudinalen und transversalen Wellen kann man jetzt auch von den in den beiden vorhergegangenen Paragraphen behandelten speziellen Fällen der Systeme 1 a, b) zu den Systemen 1 a, b) selbst über-

gehen. Die in § 2 defininierten Funktionen u_1, u_2, u_3, u_4 genügen nämlich nicht nur den Wellengleichungen 12), sondern es bestehen für die Vektorkomponenten u_1', u_2', u_3' auch die Gleichungen 8), so dass man wieder longitudinale Wellen erhält. Ebenso genügen die im § 3 definierten Funktionen u_x'', u_y'', u_x'' den Wellengleichungen 6) und der Gleichung 7), so dass durch diese Vektorkomponenten transversale Wellen gegeben sind. Wir können nun diese beiden Wellenbewegungen zusammensetzen und erhalten so die neuen Komponenten

$$
16) \qquad
\begin{aligned}
u_1 &= \frac{\partial \varphi_4}{\partial x_1} + \frac{\partial \varphi_3}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_3} \\
u_2 &= \frac{\partial \varphi_4}{\partial x_2} + \frac{\partial \varphi_1}{\partial x_3} - \frac{\partial \varphi_3}{\partial x_1} \\
u_3 &= \frac{\partial \varphi_4}{\partial x_3} + \frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} \\
u_4 &= \frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} + \frac{\partial \varphi_3}{\partial x_3},
\end{aligned}
$$

wobei zu beachten ist, dass curl $\mathfrak{u}'$ und div $\mathfrak{v}''$ verschwinden und die in Betracht kommenden Gleichungen des § 2 unverändert bleiben, wenn man zu φ_4' noch die von t unabhängige Potentialfunktion φ_4'' hinzufügt. Setzt man also longitudinale und transversale Wellen zusammen, so dass die Gleichungen 12) ohne Hinzutreten der Bedingungsgleichungen 7) oder 8) bestehen, so kann man auch mittels der Gleichungen 16) die Funktionen u_1, u_2, u_3, u_4 bilden. Aus den Gleichungen 12) und 16) ergibt sich dann unmittelbar das System 1a), während man das System 16) aus den Gleichungen 16) durch Differentiation erhält.

Nach dieser Einführung der Systeme 1 a, b) handelt es sich noch darum, im Anschluss an die vorausgegangenen Betrachtungen auch ihre Bedeutung anzugeben. Wie im II Teile der ersten der oben citierten Arbeiten des Verfassers durchgeführt wurde, kann man die Systeme 1 a b) durch eine einfache Transformation in zwei solche symmetrische System verwandeln, dass das eine durch Vertauschung der Buchstaben u und v in das andere übergeht; es genügt daher die Bedeutung des einen, z. B. 1 a), zu untersuchen. Berücksichtigt man die Gleichungen 16), so sieht man unmittelbar, dass das System 1a) nur eine andere Schreibweise der Gleichungen 12) ist. Man kann daher auch das System 16) in dieser Weise interpretieren; es ist dadurch auch wieder eine Wellenbewegung ge-

geben, deren Zusammenhang mit der ursprünglich gegebenen sich aus den Gleichungen 16) ergibt.

§ 5. Das allgemeine zeitlich veränderliche Vektorfeld in einem homogenen isotropen Medium.

Von der Betrachtung von Wellenbewegungen wollen wir jetzt zur Untersuchung eines beliebigen zeitlich veränderlichen Vektorfeldes übergehen, von dem wir nur voraussetzen, dass das betreffende Medium (der Träger des Vektorfeldes) homogen und isotrop sei und die betrachteten Funktionen innerhalb ihres Geltungsbereiches eindeutig seien und den erforderlichen Stetigkeits- und Differenzierbarkeitsbedingungen. Bei der Untersuchung eines solchen Vektorfeldes kann man davon ausgehen, dass man seine zeitlich aufeinanderfolgenden Zustände betrachtet. Dadurch gelangt man schon zu einer Einteilung dieser Vektorfelder in zwei Klassen, nämlich in solche Felder, bei denen derselbe Zustand nach einer gewissen Zeit periodisch wiederkehrt, und in solche, bei denen dies nicht der Fall ist.

Um dies mathematisch ausdrücken zu können, gehen wir in derselben Weise wie in den früheren Paragraphen vor, indem wir zuerst Zustandsänderungen betrachten, die sich längs einer einzelnen Geraden fortpflanzen. Sind diese Zustandsänderungen periodisch, so erhält man die Gleichungen 6) mit den Bedingungsgleichungen 5), sonst treten an die Stelle von 6) die Gleichungen

$$17)\qquad \begin{aligned} \frac{\partial^2 \xi}{\partial t^2} + X &= \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \\ \frac{\partial^2 \eta}{\partial t^2} + Y &= \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2} \\ \frac{\partial^2 \zeta}{\partial t^2} + Z &= \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}, \end{aligned}$$

wo X, Y, Z beliebig vorgegebene Funktionen von x, y, z und t sein sollen. Die Gleichungen 17) kann man jetzt in analoger Weise untersuchen wie früher die Gleichungen 6), da die Ausdrücke auf der rechten Seite von 6) und 17) dieselben sind und bei der Untersuchung der Gleichungen 6) auch nur von diesen Ausdrücken ausgegangen wurde. Auf diese Weise gelangt man zu nichtperiodischen Zustandsänderungen die längs einer Geraden fortschreiten und bei denen der sich ändernde Vektor parallel bzgw. senkrecht zur Fortpflanzungsrichtung ist, wofür wir der Kürze wegen auch die Bezeichnung „longitudinale“ und „transversale Bewegungen“ verwenden wollen, die Bedingungen 3) und 4) oder 7) und 8) für Longitudinalität

und Transversalität bleiben natürlich bestehen. Man kann sodann bei longitudinalen Bewegungen wieder eine skalare Funktion φ_4' einführen, die ebenfalls einer Gleichung von der Form 17) genügt, und ebenso bei transversalen Bewgungen eine von t unabhängige Potentialfunktion φ_4'', so dass man von den Gleichungen 17) für den dreidimensionalen Raum zu analogen Gleichungen für den vier dimensionalen Raum gelangt die aber nach den früheren Erörterungen den Gleichungen 17) äquivalent sind.

Von der Untersuchung von längs einer Geraden fortschreitenden Zustandsänderungen kann man sodann ebenfalls wieder zur Untersuchung der Zustandsänderungen eines Vektorfeldes in einem homogenen isotropen Medium übergehen, indem man von den Bedingungsgleichungen 5) absieht. Durch die schon früher erwähnte Schlussweise ergibt sich dann, dass man sich die Vorgänge bei einem zeitlich veränderlichen Vektorfelde in einem homogenen isotropen Medium sowohl bei longitudinalen als auch bei transversalen Bewegungen dadurch veranschaulichen kann, dass man berücksichtigt, dass sich die Zustandsänderungen nur längs der Geraden einer Strahlenkongruenz fortpflanzen, da vorausgesetzt ist, dass der Träger des Vektorfelder homogen und isotrop sei und die abhängigen Veränderlichen eindeutige Funktionen von x, y, z und t seien.

Endlich kann man ebenso wie früher wieder zur Zusammensetzung von longitudinalen und transversalen Bewegungen übergehen und gelangt so mit Verwendung der Bezeichnungsweise der Gleichungen 12) zum analogen Gleichungssystem

$$18)\qquad \frac{\partial^2 \varphi_i}{\partial t^2} + f_i = \frac{\partial^2 \varphi_i}{\partial x_1^2} + \frac{\partial^2 \varphi_i}{\partial x_2^2} + \frac{\partial^2 \varphi_i}{\partial x_3^2}, \ (i=1, 2, 3, 4).$$

zu dem jetzt keine Bedingungsgleichung hinzutritt. Da die Funktionen f_i beliebig vorgegeben werden können, enthält dieses Gleichungssystem schon den allgemeinen Fall eines zeitlich veränderlichen Vektorfeldes in einem homogenen isotropen Medium, denn es sagt nicht mehr aus, als dass die Funktionen φ_i abgesehen von den entsprechenden Stetigkeits- und Differenzierbarkeitsbedingungen entweder der Wellengleichung genügen oder nicht, je nachdem die Funktionen f_i verschwinden oder nicht. Durch die Aufstellung des Gleichungssystems 18) haben wir also auch schon einen mathematischen Ausdruck für die Einteilung der zeitlich veränderlichen Vektorfelder in die oben angegebenen beiden Klassen erhalten.

Da die Gleichungen 17) jetzt auch noch dieselbe Bedeutung behalten wie früher, so kann man durch die Zusammensetzung von longitudinalen und transversalen Bewegungen auch diese Gleichungen einführen und ge-

langt dann zum Gleichungssystem 16), während sich aus den Systemen 18) und 17) die Gleichungen ergeben

$$
\begin{aligned}
&\frac{\partial u_4}{\partial x_1}-\frac{\partial v_1}{\partial t}+\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}=f_1\\
&\frac{\partial u_4}{\partial x_2}-\frac{\partial v_2}{\partial t}+\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}=f_2\\
1\,c)\qquad &\frac{\partial u_4}{\partial x_3}-\frac{\partial v_3}{\partial t}+\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}=f_3\\
&\frac{\partial v_4}{\partial t}+\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=f_4
\end{aligned}
$$

die jetzt an die Stelle der Gleichungen 1 a) treten, die Funktionen u_1, u_2, u_3, u_4 genügen daher jetzt nicht mehr der Wellengleichung.

Nachdem wir so zum allgemeinen Fall der zeitlich veränderlichen Vektorfelder in homogenen isotropen Medien gelangt sind, wollen wir noch kurz zwei Grenzfälle betrachten. Ein naheliegender Grenzfall ist das stationäre Vektorfeld, für das die Differentialquotienten von u_i und v_i nach der Zeit verschwinden; wir wollen aber die beiden Fälle, wo die Bedingung für Transversalität und die für Longitudinalität erfüllt ist, getrennt anführen. Im ersten Falle erhält man dieselben Gleichungen wie für zeitlich unveränderliche elektrische und magnetische Felder; ebenso erhält man im zweiten Falle durch die Einführung der Funktion φ_4' analoge Gleichungen der Potentialtheorie. Natürlich versagt jetzt die hier angegebene Methode der Veranschaulichung eines zeitlich veränderlichen Vektorfeldes; an ihre Stelle tritt die in der Potentialtheorie übliche Veranschaulichung eines Vektorfeldes durch Niveaulinien und Kraftlinien.

Ein anderer Grenzfall ergibt sich aus den Gleichungen 17), wenn in ihnen die Ausdrücke auf der rechten Seite verschwinden. Dies ist bei starren Körpern der Fall, der bei ihnen keine Wellenbewegung möglich ist, so dass die Funktionen u_1, u_2, u_3, u_4 und v_4 verschwinden und daher auch die Ausdrücke auf der rechten Seite von 17). Man erhält dann die Grundgleichungen der Mechanik; allerdings sind sie jetzt nur ein spezieller Fall der Grundgleichungen der Theorie der Vektorfelder, nämlich die Grundgleichungen für ein bei der Bewegung eines starren Körpers sich ergebendes Vektorfeld, enthalten aber als Grenzfall die Bewegungsgleichungen eines Punktes. Das Verschwinden der Ausdrücke auf der rechten Seite von 17) ist natürlich auch für elastische und flüssige Medien möglich; dies trifft zu, wenn sie sich bei ihrer Bewegung wie starre Körper verhalten.

§ 6. Der Energie-Impulssatz.

Im Anschluss an die Gleichungen 1 *b, c*) können wir jetzt für zeitlich veränderliche Vektorfelder in homogenen isotropen Medien den Energie-Impulssatz einführen, indem wir die einzelnen Gleichungen der Systeme 1 *b, c*) mit den entsprechenden Komponenten u_i und v_i multiplizieren und dann durch Addition bzgw. Subtraktion zusammenfassen. Auf diese Weise erhält man folgendes Gleichungssystem

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial x_1}(u_2^2+u_3^2+v_2^2+v_3^2-u_1^2-v_1^2-u_4^2-v_4^2)\\
&+\frac{\partial}{\partial x_2}(-u_1u_2-v_1v_2-u_3u_4-v_3v_4)\\
&+\frac{\partial}{\partial x_3}(-u_1u_3-v_1v_3+u_2u_4+v_2v_4)+\frac{\partial}{\partial t}(u_3v_2-v_3v_2+v_1v_4-u_1u_4)\\
&=u_2f_3-u_3f_2-u_1f_4-u_4f_1,
\end{aligned} \tag{19$_I$}$$

$$\begin{aligned}
&\frac{\partial}{\partial x_1}(-u_1u_2-v_1v_2-u_3u_4+v_2v_4)+\frac{1}{2}\frac{\partial}{\partial x_2}(u_1^2+u_3^2+v_1^2+v_3^2\\
&\qquad-u_2^2-v_2^2-u_4^2-v_4^2)\\
&+\frac{\partial}{\partial x_3}(-u_2u_3-v_2v_3-u_1u_4-v_2v_4)+\frac{\partial}{\partial t}(u_1v_3-v_1u_3+v_2u_4-u_2v_4)\\
&=u_3f_1-u_1f_3-u_2f_4-u_4f_2,
\end{aligned} \tag{19$_{II}$}$$

$$\begin{aligned}
&\frac{\partial}{\partial x_1}(-u_1u_3-v_1v_3-u_2u_4-v_2x_4)\\
&+\frac{\partial}{\partial x_2}(-u_2u_3-v_2v_3-u_1u_4+v_1v_4)\\
&+\frac{1}{2}\frac{\partial}{\partial x_3}(u_1^2+u_2^2+v^2_1+v_2^2-u_3^2-v_3^2-u_4^2-v_4^2)\\
&+\frac{\partial}{\partial t}(u_2v_1-v_2u_1+v_3u_4-u_3v_4)\\
&=u_1f_2-v_2f_1-v_3f_2-v_4f_3,
\end{aligned} \tag{19$_{III}$}$$

$$\begin{aligned}
&\frac{\partial}{\partial x_1}(u_3v_2-v_3u_2+u_1v_4-v_1u_4)+\frac{\partial}{\partial x_2}(u_1v_3-v_1u_3+u_2v_4-v_2u_4)\\
&+\frac{\partial}{\partial x_3}(u_2v_1-v_2u_1+u_3v_4-v_3u_4)
\end{aligned} \tag{19$_{IV}$}$$

$$+\frac{1}{2}\frac{\partial}{\partial t}(u_1^2+u_2^2+u_3^2+u_4^2+v_1^2+v_2^2+v_3^2+v_4^2)$$

$$=v_4 f_4-v_1 f_1-v_2 f_2-v_3 f_3.$$

Aus dieser Einführung des Energie-Impulssatzes sieht man, dass er für beliebige zeitlich veränderliche Vektorfelder in homogenen isotropen Medien gilt, da für solche Vektorfelder auch die Gleichungen 1 *b*, *c*) bestehen; es ist daher nicht notwendig, den Energie-Impulsatz für die Elektrodynamik und für die Mechanik getrennt einzuführen.

Den Prozess, durch den man von den Gleichungen 16*c*) zu den Gleichungen 19) gelangt, kann man in einfachen Weise durch Verwendung von Quaternionen und Biquaternionen darstellen. Die Gleichungen 1 *b*, *c*) kann man nämlich in der bekannten Form schreiben

$$20)\qquad j_1\frac{\partial(\mathfrak{U}+\mathfrak{V}_i)}{\partial x_1}+j_2\frac{\partial(\mathfrak{U}+\mathfrak{V}_i)}{\partial x_2}+j_3\frac{\partial(\mathfrak{U}+\mathfrak{V}_i)}{\partial x_3}+j_4\frac{\partial(\mathfrak{U}+\mathfrak{V}_i)}{\partial t}$$

$$=\mathfrak{f}+\mathfrak{g}i,\ (i=\sqrt{-1}),$$

wo $\mathfrak{U}$ und $\mathfrak{V}$ die schon in der Einleitung eingeführten Quaternionen sind, $\mathfrak{f}$ die Quaternion mit den in den Gleichungen 1*c*) eingeführten Komponenten $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, $-\mathfrak{f}_4$ ist und die Quaternion $\mathfrak{g}$ mit den Komponenten $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$, $\mathfrak{g}_4$ wegen des Bestehens der Gleichungen 16) zwar verschwindet, aber zur Darstellung der betreffenden Ausdrücke auf der linken Seite der Gleichungen 16) und 20) dienen soll; ebenso kann man die Quaternion $\mathfrak{f}$ zugleich auch zur Darstellung der entsprechenden Ausdrücke auf der linken Seite von 20) verwenden. Bildet man nun das Produkt

$$21)\qquad \{(\mathfrak{f}_1 j_1+\mathfrak{f}_2 j_2+\mathfrak{f}_3 j+\mathfrak{f}_4)+i(\mathfrak{g}_1 j_1+\mathfrak{g}_2 j_2+\mathfrak{g}_3 j_3-\mathfrak{g}_4)\}(-\mathfrak{U}+\mathfrak{V}_i),$$

so geben die 4 Ausdrücke, die nicht Koeffizienten von i sind, gerade die Ausdrücke auf der linken Seite der 4 Gleichungen 19).

Setzt man in den Gleichungen 19)

$$u_4=v_4=0,$$

so gelangt man zum Energie-Impulssatz mit dem symmetrischen Energie-Impulstensor. Man gelangt dazu aber auch durch die etwas allgemeinere Annahme

$$22)\qquad u_4=\text{const.},\ v_4=\text{const.};$$

dann fallen in den Gleichungeu 1 *b*, *c*) auch die Differentialquotienten von u_4 und v_4 weg. Zu diesem Resultat kann man aber auch gelangen, in-

dem man berücksichtigt, dass sich unter der Annahme 22) aus den Maxwellschen Gleichungen von der Form 15 a, b) folgendes davon verschiedene neue Systeme von Maxwellschen Gleichungen ergibt:

$$\frac{\partial}{\partial t}(v_1u_4-u_1v_4)=\frac{\partial}{\partial x_2}(u_3u_4+v_3v_4)-\frac{\partial}{\partial x_3}(u_2u_4+v_2v_4),$$

$$\frac{\partial}{\partial t}(v_2u_4-u_2v_4)=\frac{\partial}{\partial x_3}(u_1u_4+v_1v_4)-\frac{\partial}{\partial x_1}(u_3u_4+v_3v_4),$$

$$\frac{\partial}{\partial t}(v_3u_4-u_3v_4)=\frac{\partial}{\partial x_1}(u_2u_4+v_2v_4)-\frac{\partial}{\partial x_2}(u_1u_4+v_1v_4),$$

$$\frac{\partial}{\partial x_1}(v_1u_4-u_1v_4)+\frac{\partial}{\partial x_2}(v_2u_4+u_2v_4)+\frac{\partial}{\partial x_3}(v_3u_4+u_3v_4)=0,$$

$$\frac{\partial}{\partial t}(u_1u_4+v_1v_4)=\frac{\partial}{\partial x_3}(v_2u_4-u_2v_4)-\frac{\partial}{\partial x_2}(u_3u_4-v_2v_4),$$

$$\frac{\partial}{\partial t}(u_2u_4+v_2v_4)=\frac{\partial}{\partial x_1}(v_3u_4+u_3v_4)-\frac{\partial}{\partial x_3}(v_1u_4-u_1v_4),$$

$$\frac{\partial}{\partial t}(u_3u_4+v_3v_4)=\frac{\partial}{\partial x_2}(v_1u_4-u_1v_4)-\frac{\partial}{\partial x_1}(v_2u_4-u_2v_4),$$

$$\frac{\partial}{\partial x_1}(u_1u_4+v_1v_4)+\frac{\partial}{\partial x_2}(u_2u_4+v_2v_4)+\frac{\partial}{\partial x_3}(u_2u_4+v_2v_4)=0.$$

Vergleicht man die Ausdrücke im ersten dieser beiden Gleichungssysteme mit denen auf der linken Seite der Gleichungen 19), so sieht man wieder, dass unter der Annahme 22) in den Gleichungen 19) der symmetrische Energie-Impulstensor auftritt.

Der spezielle Fall der Gleichungen 19), wo auf der linken Seite der symmetrische Energie-Impulstensor auftritt, ist noch mit Rücksicht auf die im § 2 über das Gleichungssystem 13a') gemachten Bemerkungen von Interesse. In diesem speziellen Falle ist nämlich die linke Seite der Gleichungen 19) von derselben Form wie die linke Seite der Gleichungen 13a') und die Bedingung 11), also hier die Bedingung

$$g_{11}+g_{22}+g_{33}=g_{44},$$

ist auch erfüllt. Setzt man daher $\mathfrak{f}_i=0$, so folgt aus den Gleichungen 19) in dem speziellen Falle, wo u_4 und v_4 verschwinden, dass dann longitudinale Wellen gegeben sind.

Ein anderer bekannter spezieller Fall des in den Gleichungen 19) auftretenden Tensors ergibt sich, wenn man

$$u_1 = u_2 = u_3 = u_4 = 0$$

oder

$$v_1 = v_2 = u_3 = u_4 = 0$$

setzt; es ist dies jener Tensor, der bei der Drehung eines starren Körpers um einen festen Punkt auftritt.

§ 7. Übergang zur Riemann-Weylschen Geometrie.

Wenn wir auch von den Gleichungen 6) zu 12) und von den Gleichungen 17) zu 18) übergegangen sind, so konnten die Gleichungssysteme 12) und 18) doch durch zeitlich veränderliche Vektorfelder im dreidimensionalen Raume veranschaulicht werden, obwohl in ihnen neben den Vektorkomponenten $\varphi_1, \varphi_2, \varphi_3$ noch die skalare Funktion φ_4 auftritt. Man kann aber die Funktionen $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ zumal sie Funktionen von 4 unabhängigen Veränderlichen sind, auch als die Koordinaten von Punkten in einem vierdimensionalen Raume interpretieren. Von dieser Interpretation wird schon Gebrauch gemacht, wenn man die Differentialquotienten von $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ nach der Zeit als die Koeffizienten einer Quaternion $\mathfrak{V}$ zusammenfasst, wie dies bei der funktionentheoretischen Einführung der Gleichungen 1 *a*, *b*) in der ersten der oben citierten Arbeiten des Verfassers geschehen ist. Dadurch, dass man $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ als Koordinaten der Punkte eines vierdimensionalen Raumes interpretiert, wird nun einem zeitlich veränderlichen Vektorfelde im dreidimensionalen Raume in umkehrbar eindeutiger Weise ein Bereich in einem vierdimensionalen Raume zugeordnet, der im allgemeinen ein nichteuklidischer ist; allerdings tritt mit Rücksicht auf das Frühere hier noch die Einschränkung hinzu, dass der Träger des Vektorfeldes als homogen und isotrope vorausgesetzt wurde. Es entsteht so die Aufgabe, zu untersuchen, welche Eigenschaften eines zeitlich veränderlichen Vektorfeldes im dreidimensionalen Raume geometrischen Eigenschaften eines vierdimensionalen Raumes entsprechen.

Die geometrischen Eigenschaften eines sehr allgemeinen Bedingungen genügenden vierdimensionalen Raumes hat nun im Anschluss an Riemann H. Weyl[1] untersucht. Er gelangt dabei zu denselben mathematischen Formeln, wie sie für Gravitationsfelder und elektromagnetische Felder bestehen, und zwar so, dass diese beiden bisher immer von einander getrennt erscheinenden Felder in einheitlicher Weise zusammengefasst

1) Sitzungsber der preuss. Akad. d. Wiss. 1918.

werden; insbesondere erhält man ein reines „Gravitationsfeld," wenn die Funktionen f_{ik}, welche in derselben Weise wie im § 2 dieser Arbeit definiert sind, verschwinden. Überträgt man dieses Resultat auf die Theorie der zeitlich veränderlichen Vektorfelder, so folgt daraus, dass ein reines „Gravitationsfeld" einem solchen zeitlich veränderlichen Vektorfelde entspricht, in dem nur noch longitudinale Bewegungen auftreten können.

Ähnliche Folgerungen ergeben sich übrigens auch unabhängig von der Nebeneinanderstellung der Riemann-Weylschen Geometrie aus der Theorie der zeitlich veränderlichen Vektorfelder selbst. Es zeigt nämlich schon die Vergleichung der Maxwellschen Gleichungen mit den auch durch allgemeine funktionentheoretische Betrachtungen sich ergebenden Gleichungen 1 *a*, *b*) und den Gleichungen 1 *b*, *c*), dass zum elektromagnetischen Felde noch ein anderes Feld hinzutreten muss, wenn nicht die eigenartige Tatsache bestehen soll, dass dann kein allgemeines Vektorfeld existiere, obwohl schon in der Statik drei physikalisch voneinander verschiedene Arten von Vektorfeldern auftreten ebenso wie zufolge der Ausführungen gegen Ende des § 5 dieser Arbeit bei einem Vektorfeld, bei dem die Funktionen u_i und v_i von t unabhängig sind. Bieten diese Überlegungen auch Anhaltspunkte zu einer physikalischen Gravitationstheorie, so ergeben sich aber experimentell prüfbare Grundlagen für eine physikalische Gravitationstheorie wohl erst, wenn man von der schon fortgeschrittenen Einsteinschen Gravitationstheorie ausgeht, indem man die Eigenschaften eines zeitlich veränderlichen Vektorfeldes im dreidimensionalen Raume aufsucht, das einem solchen vierdimensionalen Raume entspricht, der in der allgemeinen Relativitätstheorie als „reines Gravitationsfeld" bezeichnet wird.

Als ein Beitrag zu einer solchen Untersuchung erscheint schon die Gegenüberstellung der in der Einsteinschen Gravitationstheorie sich ergebenden Ablenkung der Lichtstrahlen im Gravitationsfeld der Sonne und die im § 2 dieser Arbeit erhaltene Ablenkung von transversalen Wellen in einem Potentialfelde; es ist dies zugleich auch ein erstes Beispiel einer Beziehung zwischen einem Gravitationsfelde und einem elektromagnetischen Felde.

Ein anderes einfaches, aber doch wichtiges Resutat, das sich aus der Gegenüberstellung der Riemann-Weylschen Geometrie und der Theorie der zeitlich veränderlichen Vektorfelder ergibt, enthält die in der oben angeführten Arbeit von H. Weyl gemachte Bemerkung, dass der euklidische vierdimensionale Raum ein zugleich „elektrizitäts- und gravitationsfreier" Raum ist. Es entspricht daher dem vierdimensionalen

euklidischen Raume als zeitlich veränderliches Vektorfeld das Feld eines sich bewegenden starren Körpers; somit entspricht der Ablehnung des starren Körpers in der Mechanik auch die Ablehnung des vierdimensionalen euklidischen Raumes in der Geometrie.

§ 8. Relativitätstheoretische Betrachtungen.

Wir wollen jetzt noch zeigen, dass die Gleichungen $1b, c)$ und die Gleichungen 19) von der Wahl des Koordinatensystems unabhängig sind. Dabei gehen wir davon aus, dass man die Gleichungen $1\ b, c)$ in der Form 20) schreiben kann. Berücksichtigt man, dass auch durch die unabhängigen Veränderlichen x_1, x_2, x_3, t ein Vektor im vierdimensionalen Raume gegeben ist, so dass

$$x_1 = r\alpha_1, \quad x_2 = r\alpha_2, \quad x_3 = r\alpha_3, \quad t = r\alpha_4$$

ist, wo α_1, α_2, α_3, α_4 die Richtungscosinus dieses Vektors im vierdimensionalen Raume sind, so sieht man, dass die linke Seite der Gleichung 20) als Produkt einer Quaternion mit einer Biquaternion aufgefasst werden kann, da sich dieser Ausdruck in der Form

$$(\alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3 + \alpha_4)\frac{\partial}{\partial r}(\mathfrak{U} + \mathfrak{B}_i)$$

schreiben lässt; diesen Ausdruck kann man nun auf das Produkt von Quaternionen zurückführen. Da aber dem Produkte von zwei Quaternionen die Zusammensetzung von zwei Drehstreckungen(¹) entspricht, ist es unabhängig von der Wahl des Koordinatenssystems. Daraus ergibt sich schon, dass die Gleichungen $1\ b, c)$ von der Wahl des Koordinatensystems unabhängig sind, da die Funktionen f_i Komponenten eines Vierervektors sind, der also auch wieder von der Wahl des Koordinatensystems unabhängig ist. Durch Anwendung derselben Schlussweise kann man auch zeigen, dass die Gleichung 21) und somit auch das System 19) von der Wahl des Koordinatensystems unabhängig sind. Es genügen also die Grundgleichungen $1\ b, c)$ der Theorie der Vektorfelder und die daraus sich ergebenden Gleichungen 19) den Forderungen der Invarianz im Sinne der allgemeinen Relativitätstheorie. Dies Forderung der Invarianz ist aber vom geometrischen Standpunkte aus leicht verständlich; sie hat für den vierdimensionalen Raum dieselbe Bedeutung wie in der Invariantentheorie der binären, ternären und quaternären Formen enthaltene Tatsache, dass

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(¹) Klein u. Sommerfeld, Theorie des Kreisels, Bd 1, Leipzig 1897, p. 22.

die geometrischen Eigenschaften geometrischer Gebilde von der Wahl des Koordinatensystems unabhängig sind und nur solche Beziehungen als geometrische Eigenschaften von Gebilden angesehen werden können, die von der Wahl des Koordinatensystems unabhängig sind.

Bei diesem Beweise der Unabhängigkeit der Gleichungen 1 *b*, *c*) und 19) von der Wahl des Koordinatensystems haben wir die in diesen Gleichungen auftretenden Veränderlichen als Vektoren im vierdimensionalen Raume angesehen. Wir können diese Gleichungssysteme aber auch wieder als Gleichungen betrachten, die sich auf ein zeitlich veränderliches Vektorfeld im dreidimensionalen Raume beziehen; es müssen dann diese Gleichungen die Eigenschaft haben, dass sie davon unabhängig sind, in welcher Art sich das betreffende Vektorfeld im dreidimensionalen Raume bewege. Der einfachste Fall ist dann der einer gleichförmigen Translation des betreffenden Vektorfeldes; man gelangt dann zur Problemstellung der speziellen Relativitätstheorie.

Überblicken wir die erhaltenen Resultate, so sehen wir, dass die hier untersuchten mathematischen Formeln auf drei verschiedenen Gebieten interpretiert werden können, nämlich in der Riemann-Weylschen Geometrie des vierdimensionalen Raumes, in der Theorie der zeitlich veränderlichen Vektorfelder und, wie dies in der ersten der anfangs zitierten Arbeiten des Verfassers durchgeführt wurde, durch Verallgemeinerung des Begriffes der analytischen Funktion einer komplexen Veränderlichen, indem man von der Analysis der gemeinen komplexen Zahlen zur Analysis der Hamiltonschen Quaternionen und Biquaternionen übergeht.

On the Maximum and Minimum Values of Shortest Distance between Two Screws,

by

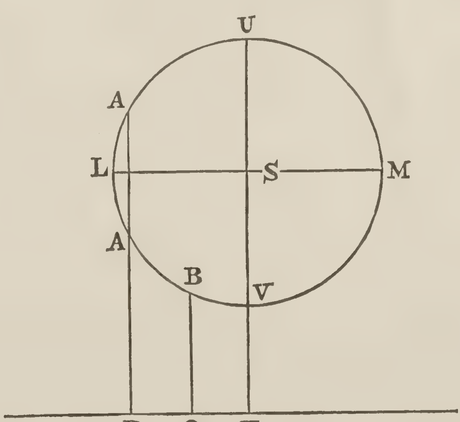
PANDIT OUDH UPADHYAYA, Calcutta, India.

Let $ABVU$ be the representative circle of the cylindroid and let PT be the axis of the pitch. Then it is evident that AP is the pitch of the screw. It is always convenient to refer to a screw as simply equivalent to its corresponding point on the circle. Thus in the figure given above, the two points A and B may conveniently be called the screws A and B. As Sir R. S. Ball has shown, it is found that every thing about a screw can be ascertained from the position of its corresponding point on the circle. Sir. R. S. Ball has proved that the shortest distance between two screws A and B is equal to the projection of ths chord AB on the axis of the pitch [Theory of Screw by R. S. Ball, page 48].

So far as I am aware, the range of the shortest distance has not been determined by any previous writer. I have not considered the negative value of the projection.

The object of this paper is to find the maximum and minimum values of the shortest distance between two screws.

Let us suppose at first that both the screws, represented by A and B, coincide with V and then the projection of AB reduces to zero. Hence the minimum value of the shortest distance between two screws is zero.



Now let us suppose that the screw B coincides with V and the screw A begins to move farther and farther towards the left. Let us suppose that A comes and ultimately coincides with B; then the projection of BV is the shortest distance between the two screws and which evidently is equal to QT in this case. Thus it is at once evident that the shortest distance between them begins to increase gradually. As the screw travels from V towards L, the shortest distance goes on increasing and when A coincides with L, the magnitude of the shortest distance becomes equal to the radius of the circle, and then again it begins to decrease and ultimately becomes zero when A coincides with V. After that it begins to increase again and becomes equal to the radius of the circle when A coincides with M. After that, it begins to decrease and becomes zero when A coincides with V. Thus, it is clear that, when one of the screws coincides with V and remains coincident with it, while the other is in any other possible position, the greatest value of the shortest distance is equal to the radius of the circle. Similar will be the case if A coincides with V and B occupies any other position.

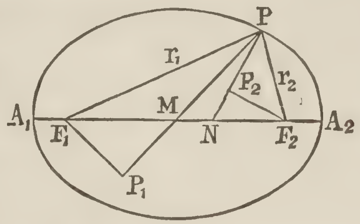
Let us now suppose that both of them begin to move, one to the right and the other to the left. Then, when one coincides with L and the other with M, the shortest distance between them will be equal to LM, because the projection of LM on PT will be equal to LM, which is the diameter of the circle. In all other cases the value of the projection will be less than that. Hence the greatest value of the shortest distance between two screws is equal to the diameter of the representative circle. Hence the range of the shortest distance between two screws varies from zero to the diameter of the representative circle.

On a Geometrical Property of the Cassinian,

by

PANDIT OUDH UPADHYAYA, Calcutta, India.

If P be any point on the Cassinian, in which the two foci are represented by F_1 and F_2, F_1P by r_1, F_2P by r_2, the middle point of F_1F_2 by M, the normal at P by PN, the perpendicular from F_1 on PM by F_1P_1, the perpendicular from F_2 on PN by F_2P_2, then I prove that $r_1F_2P_2=r_2F_1P_1$.



Let all the quantities given in the problem be denoted as in the adjoined figure. Then, it is a well-known property of the Cassinian that the angle $F_1PN= \angle F_2PM$. Therefore $\angle F_1PP_1=F_2PP_2$.

Also the angles F_1P_1P and F_2P_2P are equal, each being a right angle. Therefore $\angle P_1F_1P=P_2F_2P$.

Therefore the triangles PF_2P_1 and PF_2P_2 are similar.

Therefore
$$\frac{r_1}{r_2}=\frac{F_1P_1}{F_2P_2},$$

or
$$r_1F_2P_2=r_2F_1P_1.$$

Cyclotomic Sexe-Section for the Primes 37, 43, 67 and 73,

by

PANDIT OUDH UPADHYAYA, Calcutta, India.

The problem of sexe-section for several primes has been considered by me in papers published some time ago in this Journal. The object of this short paper is to consider the same problem for the primes 37, 43, 67 and 73.

1. Let a be a special root of

$$x^{37}-1=0,$$

then

$$a=1 \quad \text{or} \quad 1+a+a^2+a^3+a^4+\cdots\cdots+a^{36}=0.$$

Let all the special roots be divided into six groups by the following scheme: —

$$A=a+a^{27}+a^{26}+a^{36}+a^{10}+a^{11},$$

$$B=a^2+a^{17}+a^{15}+a^{35}+a^{20}+a^{22},$$

$$C=a^4+a^{34}+a^{30}+a^{33}+a^3+a^7,$$

$$D=a^8+a^{31}+a^{23}+a^{29}+a^6+a^{14},$$

$$E=a^{16}+a^{25}+a^9+a^{21}+a^{12}+a^{28},$$

and

$$F=a^{32}+a^{13}+a^{18}+a^5+a^{24}+a^{19}.$$

It is evident that:

$$A+B+C+D+E+F=(a+a^2+a^3+a^4+\cdots\cdots+a^{36})=-1.$$

By actual calculation it is found that:

$$\Sigma AB=-15.$$

Similarly

$$\Sigma ABC=28,$$

$$\Sigma ABCD=15,$$

$$\Sigma ABCDE=-38,$$

and

$$ABCDEF=-1.$$

Hence the sextic is found to be

$$\eta^6+\eta^5-15\eta^4-28\eta^3+15\eta^2+38\eta-1=0\ (^1).$$

2. Let a be a special root of

$$x^{43}-1=0,$$

then

$$a=1 \quad \text{or} \quad 1+a+a^2+a^3+a^4+\cdots\cdots+a^{42}=0.$$

Let all the special roots be divided into six groups by the following scheme :—

$$\begin{aligned}
A&=a\ \ +a^{41}+a^4\ \ +a^{35}+a^{16}+a^{11}+a^{21},\\
B&=a^3+a^{37}+a^{12}+a^{19}+a^5\ \ +a^{33}+a^{20},\\
C&=a^9+a^{25}+a^{33}+a^{14}+a^{15}+a^{13}+a^{17},\\
D&=a^{27}+a^{32}+a^{22}+a^{42}+a^2\ \ +a^{39}+a^8,\\
E&=a^{38}+a^{10}+a^{23}+a^{40}+a^6\ \ +a^{31}+a^{24},\\
F&=a^{28}+a^{30}+a^{26}+a^{34}+a^{18}+a^7\ \ +a^{29}.
\end{aligned}$$

It is evident that

$$A+B+C+D+E+F=(a+a^2+a^3+a^4+\cdots\cdots+a^{42})=-1.$$

By actual calculation it is found that:

$$\Sigma AB=4.$$

Similarly $$\Sigma ABC=23,$$

$$\Sigma ABCD=67,$$

$$\Sigma ABCDE=93,$$

and $$ABCDEF=44.$$

Hence the sextic is found to be

$$\eta^6+\eta^5+4\eta^4-23\eta^3+67\eta^2-93\eta+44=0(^1).$$

3. Let a be a special root of

$$x^{67}-1=0,$$

then $$a=1 \quad \text{or} \quad 1+a+a^2+a^3+\cdots\cdots+a^{66}=0.$$

(1) I should like to mention that I have received help in calculation from Pandit Shukdeo Chaubey.

Let all the special roots be divided into six groups by the following scheme:—

$$A = a + a^{64} + a^{9} + a^{40} + a^{14} + a^{25} + a^{59} + a^{24} + a^{62} + a^{15} + a^{22},$$
$$B = a^{2} + a^{61} + a^{18} + a^{13} + a^{28} + a^{50} + a^{51} + a^{48} + a^{57} + a^{30} + a^{44},$$
$$C = a^{4} + a^{55} + a^{36} + a^{26} + a^{56} + a^{33} + a^{35} + a^{29} + a^{47} + a^{60} + a^{21},$$
$$D = a^{8} + a^{43} + a^{5} + a^{52} + a^{45} + a^{66} + a^{3} + a^{58} + a^{27} + a^{53} + a^{42},$$
$$E = a^{16} + a^{19} + a^{10} + a^{37} + a^{23} + a^{65} + a^{6} + a^{49} + a^{54} + a^{39} + a^{17},$$

and
$$F = a^{32} + a^{38} + a^{20} + a^{7} + a^{46} + a^{63} + a^{12} + a^{31} + a^{41} + a^{11} + a^{34}.$$

Then it is evident that

$$A + B + C + D + E + F = (a + a^{2} + a^{3} + \cdots\cdots + a^{66})$$
$$= -1.$$

By calculation it is found that

$$\begin{aligned} AB = {} & 2a + a^{2} + 4a^{3} + a^{4} + 4a^{5} + 2a^{6} + a^{7} + 4a^{8} + 2a^{9} + 2a^{10} + a^{11} + a^{12} + a^{13} \\ & + 2a^{14} + 2a^{15} + 2a^{16} + 2a^{17} + a^{18} + 2a^{19} + a^{20} + a^{21} + 2a^{22} + 2a^{23} + 2a^{24} + 2a^{25} \\ & + a^{26} + 4a^{27} + a^{28} + a^{29} + a^{30} + a^{31} + a^{32} + a^{33} + a^{34} + a^{35} + a^{36} + 2a^{37} + a^{38} \\ & + 2a^{39} + 2a^{40} + a^{41} + 4a^{42} + 4a^{43} + a^{44} + a^{45} + a^{46} + a^{47} + a^{48} + 2a^{49} + a^{50} + a^{51} \\ & + 4a^{52} + 4a^{53} + 2a^{54} + a^{55} + a^{56} + a^{57} + 4a^{58} + 2a^{59} + a^{60} + a^{61} + 2a^{62} + a^{63} \\ & + 2a^{64} + 2a^{65} + 4a^{66}, \end{aligned}$$

$$\begin{aligned} BC = {} & a + 2a^{2} + a^{3} + a^{4} + a^{5} + 4a^{6} + 2a^{7} + a^{8} + a^{9} + 4a^{10} + 2a^{11} + 2a^{12} + 2a^{13} \\ & + a^{14} + a^{15} + 4a^{16} + 4a^{17} + 2a^{18} + 4a^{19} + 2a^{20} + a^{21} + a^{22} + 4a^{23} + a^{24} + a^{25} \\ & + a^{26} + a^{27} + 2a^{28} + a^{29} + 2a^{30} + 2a^{31} + 2a^{32} + a^{33} + 2a^{34} + a^{35} + a^{36} + 4a^{37} \\ & + 2a^{38} + 4a^{39} + a^{40} + 2a^{41} + a^{42} + a^{43} + 2a^{44} + a^{45} + 2a^{46} + a^{47} + 2a^{48} + 4a^{49} \\ & + 2a^{50} + 2a^{51} + a^{52} + a^{53} + 4a^{54} + a^{55} + a^{56} + 2a^{57} + a^{58} + a^{59} + a^{60} + 2a^{61} \\ & + a^{62} + 2a^{63} + a^{64} + 4a^{65} + a^{66}, \end{aligned}$$

$$\begin{aligned} CD = {} & 2a + a^{2} + a^{3} + 2a^{4} + a^{5} + a^{6} + 4a^{7} + a^{8} + 2a^{9} + a^{10} + 4a^{11} + 4a^{12} + a^{13} \\ & + 2a^{14} + 2a^{15} + a^{16} + a^{17} + a^{18} + a^{19} + 4a^{20} + 2a^{21} + 2a^{22} + a^{23} + 2a^{24} + 2a^{25} \\ & + 2a^{26} + a^{27} + a^{28} + 2a^{29} + a^{30} + 4a^{31} + 4a^{32} + 2a^{33} + 4a^{34} + 2a^{35} + 2a^{36} + a^{37} \\ & + 4a^{38} + a^{39} + 2a^{40} + 4a^{41} + a^{42} + a^{43} + a^{44} + a^{45} + 4a^{46} + 2a^{47} + a^{48} + a^{49} \\ & + a^{50} + a^{51} + a^{52} + a^{53} + a^{54} + 2a^{55} + 2a^{56} + a^{57} + a^{58} + 2a^{59} + 2a^{60} + a^{61} + 2a^{62} \\ & + 4a^{63} + 2a^{64} + a^{65} + a^{66}. \end{aligned}$$

Calculating all symmetric functions of the type AB, we find that

$$\Sigma AB = 33 + 27(a + a^{2} + a^{3} + \cdots\cdots + a^{66})$$
$$= 6.$$

By calculation it is found that

$$\begin{aligned}ABC = {} & 19a + 19a^2 + 20a^3 + 19a^4 + 20a^5 + 17a^6 + 26a^7 + 20a^8 + 19a^9 + 17a^{10} \\ & + 26a^{11} + 26a^{12} + 19a^{13} + 19a^{14} + 19a^{15} + 17a^{16} + 17a^{17} + 19a^{18} + 19a^{19} \\ & + 26a^{20} + 19a^{21} + 19a^{22} + 17a^{23} + 19a^{24} + 19a^{25} + 19a^{26} + 20a^{27} + 19a^{28} \\ & + 19a^{29} + 19a^{30} + 26a^{31} + 26a^{32} + 19a^{33} + 26a^{34} + 19a^{35} + 19a^{36} + 17a^{37} \\ & + 26a^{38} + 17a^{39} + 19a^{40} + 26a^{41} + 20a^{42} + 20a^{43} + 19a^{44} + 20a^{45} + 26a^{46} \\ & + 19a^{47} + 19a^{48} + 17a^{49} + 19a^{50} + 19a^{51} + 20a^{52} + 20a^{53} + 17a^{54} + 19a^{55} \\ & + 19a^{56} + 19a^{57} + 20a^{58} + 19a^{59} + 19a^{60} + 19a^{61} + 19a^{62} + 26a^{63} + 19a^{64} \\ & + 17a^{65} + 20a^{66} + 11.\end{aligned}$$

Calculating all symmetric functions of the type ABC, we find that

$$\begin{aligned}\Sigma ABC &= 352 + 398(a + a^2 + a^3 + \cdots\cdots + a^{66}) \\ &= -46.\end{aligned}$$

Similarly

$$\begin{aligned}\Sigma ABCD &= 3399 + 3296(a + a^2 + a^3 + \cdots\cdots a^{66}) \\ &= 123, \\ \Sigma ABCDE &= 14256 + 14425(a + a^2 + a^3 + \cdots\cdots + a^{66}) \\ &= -169, \\ ABCDEF &= 27049 + 26432(a + a^2 + a^3 + \cdots\cdots + a^{66}) \\ &= 617.\end{aligned}$$

Hence the sextic in question is found to be

$$\eta^6 + \eta^5 + 6\eta^4 + 46\eta^3 + 123\eta^2 + 169\eta + 617 = 0.$$

4. If p is a prime number and q a factor of $p-1$, there is an equation of degree q with rational coefficients, each of whose roots is the sum of $(p-1)/q$ of the primitive pth roots of unity; no such pth root occurring in more than one of the sums.

The cases for $q=2$, 3, 4 or 5 have been completed by the combined efforts of many distinguished mathematicians including A. Cayley, Miss. Charlotte Angas Scott, L. J. Rogers and W. Burnside.

I will now consider the same problem when $q=6$, for the prime 73.

Let a be a special root of

$$x^{73} - 1 = 0,$$

then $$a = 1 \quad \text{or} \quad 1 + a + a^2 + a^3 + \cdots\cdots + a^{72} = 0.$$

Let all the special roots be divided into six groups by the following scheme:—

$$A = a + a^3 + a^9 + a^{27} + a^8 + a^{24} + a^{72} + a^{70} + a^{64} + a^{46} + a^{65} + a^{49},$$

$$B = a^5 + a^{15} + a^{45} + a^{62} + a^{40} + a^{47} + a^{68} + a^{58} + a^{28} + a^{11} + a^{33} + a^{26},$$

$$C = a^{25} + a^2 + a^6 + a^{18} + a^{54} + a^{16} + a^{48} + a^{71} + a^{67} + a^{55} + a^{19} + a^{57},$$

$$D = a^{52} + a^{10} + a^{30} + a^{17} + a^{51} + a^7 + a^{21} + a^{63} + a^{43} + a^{56} + a^{22} + a^{66},$$

$$E = a^{41} + a^{50} + a^4 + a^{12} + a^{36} + a^{35} + a^{32} + a^{23} + a^{69} + a^{61} + a^{37} + a^{38},$$

$$F = a^{59} + a^{31} + a^{20} + a^{60} + a^{34} + a^{29} + a^{14} + a^{42} + a^{13} + a^{39} + a^{44} + a^{53}.$$

Then it is evident that

$$\begin{aligned} A+B+C+D+E+F &= (a + a^2 + a^3 + \cdots\cdots\cdots + a^{72}) \\ &= -1. \end{aligned}$$

By calculation it is found that

$$\begin{aligned} AB = {} & 2a + 3a^2 + 2a^3 + 3a^4 + 3a^6 + a^7 + 2a^8 + 2a^9 + a^{10} + 3a^{12} + 3a^{13} + 3a^{14} \\ & + 3a^{16} + a^{17} + 3a^{18} + 3a^{19} + 3a^{20} + a^{21} + a^{22} + 3a^{23} + 2a^{24} + 3a^{25} + 2a^{27} + 3a^{29} \\ & + a^{30} + 3a^{31} + 3a^{32} + 3a^{34} + 3a^{35} + 3a^{36} + 3a^{37} + 3a^{38} + 3a^{39} + 3a^{41} + 3a^{42} \\ & + a^{43} + 3a^{44} + 2a^{46} + 3a^{48} + 2a^{49} + 3a^{50} + a^{51} + a^{52} + 3a^{53} + 3a^{54} + 3a^{55} + a^{56} \\ & + 3a^{57} + 3a^{59} + 3a^{60} + 3a^{61} + a^{63} + 2a^{64} + 2a^{65} + a^{66} + 3a^{67} + 3a^{69} + 2a^{70} \\ & + 3a^{71} + 2a^{72}, \end{aligned}$$

$$\begin{aligned} BC = {} & 3a + 3a^3 + a^4 + 2a^5 + 3a^7 + 3a^8 + 3a^9 + 3a^{10} + 2a^{11} + a^{12} + 3a^{13} + 3a^{14} \\ & + 2a^{15} + 3a^{17} + 3a^{20} + 3a^{21} + 3a^{22} + a^{23} + 3a^{24} + 2a^{26} + 3a^{27} + 3a^{28} + 3a^{29} \\ & + 3a^{30} + 3a^{31} + a^{32} + 2a^{33} + 3a^{34} + a^{35} + a^{36} + a^{37} + a^{38} + 3a^{39} + 2a^{40} + a^{41} \\ & + 3a^{42} + 3a^{42} + 3a^{43} + 3a^{44} + 2a^{45} + 3a^{46} + 2a^{47} + 3a^{49} + a^{50} + 3a^{51} + 3a^{52} \\ & + 3a^{53} + 3a^{56} + 2a^{58} + 3a^{59} + 3a^{60} + a^{61} + 2a^{62} + 3a^{63} + 3a^{64} + 3a^{65} + 3a^{66} \\ & + 2a^{68} + a^{69} + 3a^{70} + 3a^{72}, \end{aligned}$$

$$\begin{aligned} EF = {} & a + 3a^2 + a^3 + 3a^5 + 3a^6 + 2a^7 + a^8 + a^9 + 2a^{10} + 3a^{11} + 3a^{13} + 3a^{14} + 3a^{15} \\ & + 3a^{16} + 2a^{17} + 3a^{18} + 3a^{19} + 3a^{20} + 2a^{21} + 2a^{22} + a^{24} + 3a^{25} + 3a^{26} + a^{27} \\ & + 3a^{28} + 3a^{29} + 2a^{30} + 3a^{31} + 3a^{33} + 3a^{34} + 3a^{39} + 3a^{40} + 3a^{42} + 2a^{43} + 3a^{44} \\ & + 3a^{45} + a^{46} + 3a^{47} + 3a^{48} + a^{49} + 2a^{51} + 2a^{52} + 3a^{53} + 3a^{54} + 3a^{55} + 2a^{56} \\ & + 3a^{57} + 3a^{58} + 3a^{59} + 3a^{60} + 3a^{62} + 2a^{63} + a^{64} + a^{65} + 2a^{66} + 3a^{67} + 3a^{68} \\ & + a^{70} + 3a^{71} + a^{72}. \end{aligned}$$

Calulating all the symmetric functions of the type AB, we find that

$$\begin{aligned}\Sigma AB &= 30\,(a+a^2+a^3+\cdots\cdots+a^{72}) \\ &= -30.\end{aligned}$$

By calculation it is found that

$$\begin{aligned}ABC = {} & 18a+27a^2+18a^3+27a^4+21a^5+27a^6+24a^7+18a^8+18a^9+24a^{10} \\ & +21a^{11}+27a^{12}+24a^{13}+24a^{14}+21a^{15}+27a^{16}+24a^{17}+27a^{18}+27a^{19} \\ & +24a^{20}+24a^{21}+24a^{22}+27a^{23}+18a^{24}+27a^{25}+21a^{26}+18a^{27}+21a^{28} \\ & +24a^{29}+24a^{30}+24a^{31}+27a^{32}+21a^{33}+24a^{34}+27a^{35}+27a^{36}+27a^{37} \\ & +27a^{38}+24a^{39}+21a^{40}+27a^{41}+24a^{42}+24a^{43}+24a^{44}+21a^{45}+21a^{45} \\ & +18a^{46}+21a^{47}+27a^{48}+18a^{49}+27a^{50}+24a^{51}+24a^{52}+24a^{53}+27a^{54} \\ & +27a^{55}+24a^{56}+27a^{57}+21a^{58}+24a^{59}+24a^{60}+27a^{61}+21a^{62}+24a^{63} \\ & +18a^{64}+18a^{65}+24a^{66}+27a^{67}+21a^{68}+27a^{69}+18a^{70}+27a^{71}+18a^{72} \\ & +36.\end{aligned}$$

Calculating all the symmetric functions of the type ABC, we find that

$$\begin{aligned}\Sigma ABC &= 504+473(a+a^2+a^3+\cdots\cdots\cdots\cdots+a^{72}) \\ &= 504-473 \\ &= 31.\end{aligned}$$

Similarly,

$$\begin{aligned}\Sigma ABCD &= 4464+4258(a+a^2+a^3+\cdots\cdots\cdots\cdots+a^{72}) \\ &= 4464-4258 \\ &= 206,\end{aligned}$$

$$\begin{aligned}\Sigma ABCDE &= 20304+20454(a+a^2+a^3+\cdots\cdots\cdots\cdots+a^{72}). \\ &= 20304-20454 \\ &= -150,\end{aligned}$$

and $$\begin{aligned}ABCDEF &= 40824+40905\,(a+a^2+a^3+\cdots\cdots\cdots\cdots+a^{72}). \\ &= 40824-40905 \\ &= -81.\end{aligned}$$

Hence the sextic in question is found to be

$$\eta^6+\eta^5-30\eta^4-31\eta^3+206\eta^2+150\eta-81=0.$$

Cyclotomic Quinqui-Section for the Primes 1621, 1721 and 1741,

by

PANDIT OUDH UPADHYAYA, Calcutta, India.

The problem of cyclotomic quinqui-section was attempted by A. Cayley in the Proceedings of London Mathematical Society([1]), and he considered the same problem again in a second note([2]) in the Proc. of L. M. S.; but he was not able to solve the problem completely.

The same problem was considered also by H. W. Lloyd Tanner([3]) in the Proc. of L. M. S. Miss Charlotte Angas Scott([4]) considered the problem of Quarti-Section and Quinqui-Section in the American Journal of Mathematics. She made the following remark in her paper: "The remaining term I have not succeeded in finding." She was not able to find the expressions for the constant term in the period equation.

L. J. Rogers completely solved the problem of quinqui-section in the Proc. of L. M. S., vol. 32. There he has shown that the problem of quinqui-section depends upon the solution of two diophantine equations. Fairly recently the same problem has been considered by W. Burnside in the Proc. of L. M. S., 1915. There he has shown that this problem depends upon the solution of two diophantine equations, which are as follows:—

$$12^2 p=[4p-16-25(A+B)]^2+1125(A-B)^2+450(C^2+D^2), \quad (1)$$

$$0=[4p-16-25(A+B)][A-B]+3(C^2+4CD-D^2). \quad (2)$$

The period as given by W. Burnside is this:—

$$\eta^5+\eta^4-\frac{2}{5}(p-1)\eta^3+\left[\frac{1}{3}p(A+B)-\frac{2(p-1)(2p+3)}{3\times5\times5}\right]\eta^2$$

$$+\left[\frac{p}{9}\left(\frac{p-1}{5}+A+B\right)^2-p\,AB-\frac{(p-1)^3}{5^3}\right]\eta+\frac{1}{5}p\left[\frac{1}{5.6^3}\right.$$

$$\left\{5(A+B)-\frac{4p-4}{5}\right\}^3+\frac{1}{6^2}\left\{\frac{2p-2}{5}-A-B\right\}^2$$

(1) The binomial equation $x^p-1=0$; quinqui-section, vol. 13, pp. 15, 16.

(2) The binomial equation $x^p-1=0$; quinqui-section, vol. 16, pp. 61–63.

(3) On binomial equation $x^p-1=0$; quinqui-section, vol. 18, pp. 214–234.

(4) American Journal of Mathematics, 1886, vol. 8, pp. 261–64.

$$+\frac{1}{4}(A-B)^2+\frac{1}{8}(A-B)(D^2-C^2)\Big]-\frac{(p-1)^3}{5^5}.$$

These three formulae, practically, were first given by **L. J. Rogers** with different notations, but in this paper I have adhered to the notation of **W. Burnside.** So far as I am aware the first two equations have been solved by him for the primes 11, 31, 41, 61, 61 and 71; but the period equation was not calculated by him even for all these primes.

The object of this paper is to solve the two diophantine equations for the primes 1621 1721 and 1741 and to calculate the period equations for the same.

1. Calculation for the prime 1621.

In the first equation let us substitute the value of p, supposing that $A+B=261$, then by the first equation we get:

$$[4\times 1621-16-25\times 261]^2+1125(A-B)^2+450(C^2+D^2)=144\times 1621$$

or
$$1125(A-B)^2+450(C^2+D^2)=233424-3249.$$

Now if $A-B=1$,

$$1125\times 1^2+450(C^2+D^2)=230175$$

or
$$(C^2+D^2)\,450=230175-1125$$

or
$$C^2+D^2=509$$
$$=22^2+(-5)^2.$$

Therefore
$$C=22 \text{ and } D=-5.$$

And because $A+B=261$ and $A-B=1$,

$$A=131,\ B=130.$$

Substituting these values in the second equation we get

$$[4\times 1621-16-25\times 261][1]+3\{22^2+4\times 22\times(-5)-(-5)^2\}=0.$$

Thus it is clear that these values ($A=131$, $B=130$, $C=22$ and $D=-5$) satisfy both the equations.

Substituting these values in the third equation we find the period equation to be

$$\eta^5+\eta^4-648\,\eta^3+843\,\eta^2+20671\,\eta+30037=0.$$

2. Calculation for the prime 1721.

With the help of the first and second equations we find that

$$A=130, \ B=141, \ C=15 \text{ and } D=2.$$

Then by substituting these values in the third equation, the period equation is found to be

$$\eta^5+\eta^4-688\,\eta^3-2547\,\eta^2+17281511\,\eta+146323=0.$$

3. Calculation for the prime 1741.

Solving the first two equations we find that

$$A=131, \ B=142, \ C=14 \text{ and } D=5.$$

By substituting these values in the third equation we find the period equation to be

$$\eta^5+\eta^4-696\,\eta^3-3273\,\eta^2+69835\,\eta+130931=0.$$

prime	A	B	C	D	Coeff. of η^5	Coeff. of η^4	Coeff. of η^3	Coeff. of η^2	Coeff. of η	Constant term
1621	131	130	22	−5	1	1	−648	843	20671	30037
1721	130	141	15	2	1	1	−688	−2547	17281511	146323
1741	131	142	14	5	1	1	−696	−3273	69835	130931

Some General Formulae in Symmetric Functions which depend upon Cyclotomic Tri-Section,

by

PANDIT OUDH UPADHYAYA, Calcutta, India.

The object of this paper is to determine some general formulae in symmetric function which depend upon cyclotomic trisection.

Let it be supposed that η_0, η_1 and η_2 are the roots of the period equation

$$\eta^3+\eta^2-\frac{p-1}{3}\eta-\frac{1}{9}\left(pa'+\frac{p-1}{3}\right)=0,$$

where p is a prime of the form $6n+1$ and a' is that integral value which is obtained by equating $3a'-2$ to that value of $\sqrt{a}$ which is obtained by representing $4p$ in the form a^2+3b^2.

Thus it is clear that

$$\Sigma\eta_0=-1,\ \Sigma\eta_0\eta_1=-\frac{p-1}{3}$$

and

$$\eta_0\eta_1\eta_2=\frac{1}{9}\left(pa'+\frac{p-1}{3}\right).$$

Case I.
$$\Sigma\eta^2=(\Sigma\eta_0)^2-\Sigma 2\eta_0\eta_1$$
$$=(-1)^2-\left(-\frac{2(p-1)}{3}\right)$$
$$=\frac{2p+1}{3}.$$

Case II.
$$\Sigma\eta_0^3=(\Sigma\eta_0)(\Sigma\eta_0^2-\Sigma\eta_0\eta_1)+3\eta_0\eta_1\eta_2$$
$$=-1\left(\frac{2p+1}{3}+\frac{p-1}{3}\right)+\frac{1}{3}\left(pa'+\frac{p-1}{3}\right)$$
$$=-p+\frac{1}{3}\left(pa'+\frac{p-1}{3}\right).$$

Case III. $\Sigma \eta_0^2\eta_1 = \eta_0^2(\eta_1+\eta_2)+\eta_1^2(\eta_2+\eta_0)+\eta_2^2(\eta_0+\eta_1)$

$$= \eta_0^2(-1-\eta_0)+\eta_1^2(-1-\eta_1)+\eta_2^2(-1-\eta_2)$$

$$= -(\Sigma \eta_0^3 + \Sigma \eta_0^2)$$

$$= p - \frac{1}{3}\left(pa' + \frac{p-1}{3}\right) - \frac{2p+1}{3}$$

$$= -\frac{1}{3}\left(pa' - \frac{2p-2}{3}\right).$$

Case IV. $\Sigma \eta_0^2\eta_1\eta_2 = \eta_0\eta_1\eta_2(\Sigma \eta_0)$

$$= \left\{\frac{1}{9}\left(pa' + \frac{p-1}{3}\right)\right\}\{-1\}$$

$$= -\frac{1}{9}\left(pa' + \frac{p-1}{3}\right).$$

Case V. $\Sigma \eta_0^2\eta_1^2 = (\Sigma \eta_0\eta_1)^2 - \Sigma 2\eta_0^2\eta_1\eta_2$

$$= \left(-\frac{p-1}{3}\right)^2 + \frac{2}{9}\left(pa' + \frac{p-1}{3}\right)$$

$$= \frac{1}{9}\left\{(p-1)^2 + 2\left(pa' + \frac{p-1}{3}\right)\right\}.$$

Case VI. $\Sigma \eta_0^2\eta_1\eta_2^2 = \eta_0\eta_1\eta_2(\Sigma \eta_0\eta_1)$

$$= \frac{1}{9}\left(pa' + \frac{p-1}{3}\right)\left(-\frac{p-1}{3}\right)$$

$$= -\frac{p-1}{27}\left(pa' + \frac{p-1}{3}\right)^{(1)}.$$

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(1) I should like to mention that I have received very much help from Pandit Shukdeo Chaubey.

The Extremal Chords of an Oval,

by

TSURUICHI HAYASHI, Sendai.

For an oval there is one and only one maximum chord in a given direction. The length of this maximum chord varies as the direction varies. This note aims to determine analytically the positions of the extrema of such a maximum chord and to prove some inequalities relating to their lengths.

1. The cartesian coordinates (x, y) and the polar tangential coordinates (p, φ) of a point on the given oval are connected by the relations

$$x \sin\varphi - y \cos\varphi = p,$$

$$x \cos\varphi + y \sin\varphi = p',$$

where p' denotes the first derivative of p with respect to φ. From these relations we have

$$x = p \sin\varphi + p' \cos\varphi,$$

$$y = -p \cos\varphi + p' \sin\varphi.$$

The equation to a straight line parallel to the tangent to the oval which makes an angle equal to φ with the positive direction of the x-axis is

$$-x \sin\varphi + y \cos\varphi = P,$$

where P is the distance of the straight line from the origin of coordinates. Changing the value of P, the part of this straight line within the oval, that is the chord of the oval, changes its length. Fixing the value of φ, I will find first of all the value of P for which this chord becomes maximum.

Let (x_1, y_1) and (x_2, y_2) be the cartesian coordinates of the extremities M_1 and M_2 of such a chord respectively. Then since the cartesian coordinates of the foot F of the perpendicular dropped on the chord from the origin of coordinates are

$$(-P \sin\varphi,\ P \cos\varphi),$$

we have

$$x_1 = -P\sin\varphi + \rho_1\cos\varphi,$$
$$y_1 = P\cos\varphi + \rho_1\sin\varphi;$$
$$x_2 = -P\sin\varphi - \rho_2\cos\varphi,$$
$$y_2 = P\cos\varphi - \rho_2\sin\varphi;$$

where ρ_1 and ρ_2 stand for the lengths of FM_1 and FM_2 respectively. These coordinates may be expressed in terms of the values φ_1 and φ_2 of φ at the points M_1 and M_2: thus

$$x_1 = p(\varphi_1)\sin\varphi_1 + p'(\varphi_1)\cos\varphi_1,$$
$$y_1 = -p(\varphi_1)\cos\varphi_1 + p'(\varphi_1)\sin\varphi_1;$$
$$x_2 = p(\varphi_2)\sin\varphi_2 + p'(\varphi_2)\cos\varphi_2,$$
$$y_2 = -p(\varphi_2)\cos\varphi_2 + p'(\varphi_2)\sin\varphi_2.$$

Equating these values to each other, we get

$$\begin{aligned} P &= -p(\varphi_1)\cos(\varphi-\varphi_1) - p'(\varphi_1)\sin(\varphi-\varphi_1) \\ &= -p(\varphi_2)\cos(\varphi-\varphi_2) - p'(\varphi_2)\sin(\varphi-\varphi_2), \end{aligned}$$

and

$$\rho_1 = -p(\varphi_1)\sin(\varphi-\varphi_1) + p'(\varphi_1)\cos(\varphi-\varphi_1),$$
$$\rho_2 = p(\varphi_2)\sin(\varphi-\varphi_2) - p'(\varphi_2)\cos(\varphi-\varphi_2).$$

From these formulae, we get

$$\frac{d(\rho_1+\rho_2)}{dP} = \cot(\varphi-\varphi_2) - \cot(\varphi-\varphi_1).$$

Hence in order that $\rho_1+\rho_2$ becomes maximum, we must have

$$\varphi_2 = \varphi_1 + \pi.$$

Hence we get the theorem: *The tangents to the oval at the extremities of the maximum one of the chords having a fixed direction are parallel.* This is easily proved by adopting Fermat's principle, *by considering a slender parallelogram of which two opposite sides are parallel and equal.*

In this case

$$\begin{aligned} P &= -p(\varphi_1)\cos(\varphi-\varphi_1) - p'(\varphi_1)\sin(\varphi-\varphi_1) \\ &= p(\varphi_1+\pi)\cos(\varphi-\varphi_1) + p'(\varphi_1+\pi)\sin(\varphi-\varphi_1) \qquad (1), \end{aligned}$$

whence

$$\{p(\varphi_1)+p(\varphi_1+\pi)\}\cos(\varphi-\varphi_1)+\{p'(\varphi_1)+p'(\varphi_1+\pi)\}\sin(\varphi-\varphi_1)=0 \quad (2),$$

and

$$\rho_1+\rho_2=-\{p(\varphi_1)+p(\varphi_1+\pi)\}\sin(\varphi-\varphi_1)+\{p'(\varphi_1)+p'(\varphi_1+\pi)\}\cos(\varphi-\varphi_1) \quad (3).$$

The value of φ_1 for the maximum chord is found from (2) and then the value of P from (1).

2. Now let the value of φ change from 0 to 2π, and let us find the extremum values of $\rho_1+\rho_2$ given by (3).

Using (2) and (3), we get

$$\frac{d(\rho_1+\rho_2)}{d\varphi}=-\cot(\varphi-\varphi_1)\left\{\begin{array}{l}\{p(\varphi_1)+p(\varphi_1+\pi)\}\sin(\varphi-\varphi_1)\\ -\{p'(\varphi_1)+p'(\varphi_1+\pi)\}\cos(\varphi-\varphi_1)\end{array}\right\}.$$

So we are to distinguish two cases in order to equate this expression to zero.

Case I. $\qquad \cot(\varphi-\varphi_1)=0.$

In this case,

$$\cos(\varphi-\varphi_1)=0, \qquad \sin(\varphi-\varphi_1)=\pm 1.$$

Hence

$$\varphi-\varphi_1=(2n+1)\pi/2,$$

where n is equal to zero or an integer. But

$$|\varphi-\varphi_1|<\pi,$$

and we can assume φ_1 greater than φ. So the only admissible value of n is -1, so that

$$\varphi_1=\varphi+\frac{\pi}{2} \quad (4).$$

Hence every chord under consideration is a double-normal of the oval, i.e. the tangents at the extremities of the chord are perpendicular to the chord. This is also easily proved by adopting Fermat's principle. But to do so we must *consider a slender quadrilateral of which one pair of opposite sides and two diagonals are all equal to one another, what is not agreeable consideration.*

By virtue of (4), equation (2) becomes

$$p'(\varphi_1)+p'(\varphi_1+\pi)=0 \quad (5).$$

From these two equations (4) and (5) we get the required values of φ_1 and then φ.

φ_1 and φ thus determined, we have

$$P=p'(\varphi_1)=-p'(\varphi_1+\pi),$$

and

$$\rho_1+\rho_2=p(\varphi_1)+p(\varphi_1+\pi).$$

Case II.

$$\{p(\varphi_1)+p(\varphi_1+\pi)\}\sin(\varphi-\varphi_1)=\{p'(\varphi_1)+p'(\varphi_1+\pi)\}\cos(\varphi-\varphi_1) \quad (6).$$

Squaring and adding the equations (2) and (6), we get

$$\left.\begin{aligned} p(\varphi_1)+p(\varphi_1+\pi)&=0,\\ p'(\varphi_1)+p'(\varphi_1+\pi)&=0.\end{aligned}\right\} \quad (7).$$

In general we have the extremal values of the maximum chord $\rho_1+\rho_2$ and the corresponding values of φ_1 and then those of φ are to be found from the two equations (2) and (6). But from these two equations, equations (7) are got. Therefore φ is not determinable and so φ_1 is also quite arbitrary. Therefore from (7), we conclude that the curve is a curve of constant breadth zero.

Therefore we arrive at the theorem: *The extremal chords of an oval $p=p(\varphi)$ are double-normals of the oval, and their positions are determined by the equation*

$$p'(\varphi_1)+p'(\varphi_1+\pi)=0,$$

and their directions by

$$\varphi=\frac{\pi}{2}-\varphi_1,$$

and their distances from the origin by

$$P=p'(\varphi_1)=-p'(\varphi_1+\pi),$$

and their lengths are equal to

$$p(\varphi_1)+p(\varphi_1+\pi),$$

so that any one of the extremal chords, the tangents at its extremities and the perpendiculars dropped from the origin to these tangents form a rectangle. In the equation

$$p'(\varphi_1)+p'(\varphi_1+\pi)=0$$

gives us an infinite number of values for φ_1, a part or the whole of tde curve consists of a curve or curves of constant breadth.

3. If D, L and F be the length of the greatest chord, the total perimeter, and the area of a given oval respectively, then the inequalities

$$2D^2 \geqq LD - 4F \geqq 0 \qquad (8),$$

hold good, as I prove later.

The inequalities are remarkable, since, if this be true, we have the relation

$$\frac{1}{4}LD \geqq F \geqq \frac{1}{4}D(L-2D) \qquad (9),$$

which gives us the upper and lower limits of F for ovals whose greatest chord D and whose perimeter L are fixed. Between D and L of ovals whose greatest chord is fixed the inequalities

$$2D \leqq L \leqq \pi D$$

hold good, for the former part is true for the thinnest oval flattened along the given chord and the latter part is proved by A. Rosenthal and O. Szász[1] to be true in case the oval is a curve of constant breadth D.

Now I proceed to prove inequalities (8).

As I have already shown the tangents at the extremities of the greatest chord are parallel and perpendicular to the chord. So enclose the oval within a rectangle, one pair of whose opposite sides are these tangents and the other pair of whose opposite sides are those tangents parallel to the greatest chord, and let the distances of the latter tangents from the chord be d_1 and d_2 respectively. Then since the perimeter of the oval is not greater than the perimeter of the enclosing rectangle, we have

$$L \leqq 2(D + d_1 + d_2).$$

Moreover from this figure we have

$$F \geqq \frac{1}{2} D d_1 + \frac{1}{2} D d_2.$$

Combining these relations

$$4F \geqq D(L-2D)\,(^2) \qquad (10),$$

which is the former part of (8) and the latter part of (9).

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(1) Eine Extremaleigenschaft der Kurven konstanter Breite, Jahresbericht d. Deutschen Math.-Vereinigung, Bd. 25, 1917, pp. 278-282, [p. 279].

(2) This relation holds good if D be the length of any double-normal of the oval. It is not necessary that the normal is the greatest.

From (10)

$$2D^2 \geqq LD - 4F.$$

But it remains to be proved that $LD-4F$ is not negative.

By a Blaschke inequality([1])

$$L^2 \geqq 4\pi F,$$

the sign of equality existing in case the oval is a circle, and by the Rosenthal-Szász inequality above-cited

$$\pi D \geqq L,$$

the sign of equality existng in case the oval is a curve of constant breadth. Hence

$$\pi D^2 \geqq 4F.$$

Multiplying these inequalities together and extracting square roots of both sides, we get

$$LD \geqq 4F,$$

the sign of equality existing in case the oval is a circle. Hence $LD-4F$ is not negative([2]).

4. Let as before (p, φ) be the polar tangential coordinates of a point on the oval and let ρ be the radius of curvature at that point.

Then

$$L = \int_0^{2\pi} \frac{ds}{d\varphi} d\varphi = \int_0^{2\pi} \rho d\varphi \qquad (11),$$

and

$$2F = \int_0^{2\pi} p \frac{ds}{d\varphi} d\varphi = \int_0^{2\pi} p\rho \, d\varphi.$$

Let ρ_1 and ρ_2 be the greatest and least radii of curvature of the oval. Then by (11)

$$2\pi\rho_1 \geqq L \geqq 2\pi\rho_2 \qquad (13).$$

Again by (12)

$$\rho_1 \int_0^{2\pi} p d\varphi \geqq 2F \geqq \rho_2 \int_0^{2\pi} p d\varphi \qquad (14).$$

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(1) Kreis und Kugel, 1916, Veit and Co., Leipsic, p. 31.

(2) I express my thanks to Prof. Kubota for his kind advice on this proof.

But

$$\int_0^{2\pi} p\,d\varphi = \int_0^{2\pi} (p+p'')\,d\varphi = \int_0^{2\pi} \rho\,d\varphi \qquad (15).$$

Hence

$$2\pi\rho_1 \geqq \int_0^{2\pi} p\,d\varphi \geqq 2\pi\rho_2$$

Hence

$$\pi\rho_1^2 \geqq F \geqq \pi\rho_2^2 \qquad (16).$$

Hence by (13) and (16): *The perimeter or area of an oval lie between the perimeters or areas of the osculating circles whose radii are greatest and least respectively.* This theorem due to Hurwitz and Blaschke is found in Blaschke's "Kreis und Kugel," p. 116([1]).

By (11), (14) and (15)

$$\rho_1 L \geqq 2F \geqq \rho_2 L.$$

Compared with inequalities (9), we have

$$\rho_1 \geqq \frac{D}{2}\left(1 - 2\frac{D}{L}\right),$$

$$\rho_2 \leqq \frac{D}{2}.$$

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([1]) The same fact is true for the area contained between a given curve and its evolute, because the formula for the area is evidently

$$\frac{1}{2}\int_0^{2\pi} \rho^2\,d\varphi.$$

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